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UNIAXIAL STRENGTH OF POLYMERIC-MATRIX FIBROUS COMPOSITES PREDICTED THROUGH A HOMOGENIZATION APPROACH

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Abstract—A homogenization technique applied in conjunction with limit analysis theory allows the prediction of the macroscopic strength of fiber composites as a function of the strength properties of the phases (i.e. fiber, matrix and interface). Emphasis is placed here on the uniaxial strength, for which bounds based on static and kinematic approaches are proposed. Special attention is devoted to the influence of the matrix, which is presumed to be polymeric and complying with Drucker–Prager, Mohr–Coulomb or a parabolic type criterion. Also the limited strength of the fiber–matrix interface is accounted for through the introduction of a Mohr–Coulomb type strength criterion. Analytical equations describing the dependence of the macroscopic strength of the composite on the orientation of the applied stress with respect to the fibers are proposed. The parameters required to define the model are limited in number and possess a clear mechanical meaning. Comparisons with experimental data available in the literature prove quite satisfactory.

1. INTRODUCTION

In recent decades a tremendous amount of work has been devoted to the mathematical modelling of the behaviour of composite materials and structures, as a consequence of their increasing usage. The majority of the papers on this subject cover the elastic range; more recently also delamination and edge effects have attracted considerable attention. Less interest has been paid to the description of the nonlinear behaviour of fiber composites: this was done, for instance, by Dvorak and Bahei-el-Din (1979), who applied a modified Hill's self-consistent method to the evaluation of the overall instantaneous moduli of elastic–plastic fibrous composites; bounds on these parameters were also obtained by Teply and Dvorak (1988) through application of extremum principles to the analysis of a representative volume of composite. Initial yield surfaces of composites with perfectly elastic fibers and elastoplastic ductile matrices were computed by Pindera and Aboudi (1988), who made use of the micromechanical analyses of Aboudi (1987) on square composite cells to relate micro- and macroscopic stresses. The elastic–plastic behaviour of uni- and bidirectional laminates was modelled by Vaziri *et al.* (1991), accounting for the subsequent failure of the different plies, which are individually considered as homogeneous and transversely isotropic. All of these works make use of plasticity theory and are mainly of interest for metal matrix composites. Also the field of nonlinear elastic composites has been investigated, mainly in the recent works by Ponte Castañeda (1991) and deBotton and Ponte Castañeda (1993), who employed variational principles in order to analytically evaluate the effective properties of these materials; actually the class of modern composites that can be described using a nonlinear elastic model is rather limited.

Many of the theoretical methods proposed in the quoted references give exhaustive details regarding the stress and strain distribution within the composite and the material response under any load history. An inherent drawback to the majority of these works is the computational complexity, which unavoidably leads to the use of numerical solving procedures (e.g. the FE method). In many instances it might be required to be able to describe only the limit behaviour of the composite, and a complete stress–strain relationship might turn out to be refined but unnecessary information. A relatively simple way to obtain

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this information is by making use of limit analysis procedures, as done for example by McLaughlin (1972), Majumdar and McLaughlin (1975) and de Buhan and Taliercio (1991) for composites with phases complying with the von Mises criterion. If polymeric composites have to be dealt with, limit analysis has to be used with some care and different strength criteria must be taken into account. This was done by Taliercio (1992) assuming the Drucker–Prager criterion for the matrix material and perfect bonding between fibers and matrix; this latter assumption may prove unacceptable in several instances.

The aim of the present work is to apply the procedure used in previous papers by de Buhan and Taliercio (1991) and Taliercio (1992) to the prediction of the overall (or macroscopic) strength properties of fiber composites, based on the strength properties of the phases. Special attention is devoted to the influence of the nature of the matrix material and the fiber–matrix interface. Here, only the macroscopic uniaxial strength will be dealt with, though the method is in principle applicable to any macroscopic state of stress (i.e. a macroscopic strength domain could be defined). This method makes use of homogenization theory for periodic media applied to limit analysis, whose fundamentals will be briefly reviewed in Section 2. Although the exact macroscopic strength domain of the composite is, in general, unknown but for its theoretical definition, by means of static and kinematic limit analysis approaches inner and outer bounds on this domain can be obtained, whose definition will be recalled in Sections 2.1 and 2.2, respectively. The main features of the results yielded by these bounds will be discussed in Section 2.3. Particular strength criteria suitable to the description of the ultimate properties of polymeric matrices will be dealt with in the first part of Section 3, namely a parabolic-type strength criterion (Section 3.1), the Drucker–Prager criterion (Section 3.2) and the Mohr–Coulomb criterion (Section 3.3). A number of reasons explained in Section 3.4 lead to the introduction of a fiber–matrix interface strength criterion, which is presumed to be of the Coulomb type (Section 3.5). Analytical expressions for the bounds on the macroscopic uniaxial strength will be derived in the majority of the considered cases. In Section 4 the reliability of the proposed model will be checked through comparisons with available experimental results; it reveals to be at least as effective as the widely used Tsai–Wu criterion, in which, contrary to the present model, some of the strength parameters lack a clear physical meaning. Finally, in Section 5 some concluding remarks on the model capabilities and limits of applicability are made.

2. GENERAL FORMULATION

Fiber composites fall within the class of strongly heterogeneous media, since the presence of reinforcing fibers embedded in a bonding matrix makes their mechanical properties rapidly varying from point to point. Obviously, it is advisable to be capable of describing the mechanical properties of the composite as a whole, i.e. on a macroscopic scale. Powerful tools to this end are different *homogenization techniques*, whose common feature entails replacing the given heterogeneous medium with an "equivalent" homogeneous one. A broad literature on homogenization exists, but it is sufficient to recall here the comprehensive textbook of Sanchez-Palencia (1980), in which all the fundamental aspects of this theory are presented.

Many of the existing homogenization procedures are based on the assumption that the composite is "periodic", i.e. that the reinforcing fibers are evenly spaced and equal in cross-section; this is the assumption that will be made here, which implies leaving composites reinforced by randomly oriented short fibers out of consideration. Actually, only unidirectional composites with long fibers of circular cross-section will be dealt with here and any randomness related to fiber misalignment, strength, volume fraction, etc., will be neglected. As a consequence of the assumed periodicity, a *unit cell*, Y, containing a fiber embedded in the matrix material, suffices to characterize the considered medium. Y_m and Y_f will denote the parts of Y consisting of matrix and fiber, respectively; the fiber–matrix interface is supposed to reduce to a surface, denoted in the sequel by S_{int}. η will denote the fiber volume fraction, so that the matrix volume fraction is $1-\eta$. The fiber axis will be denoted by x; the length of the unit cell along x is apparently arbitrary and will be taken equal to 1. Unit cells of two different geometries will be studied, corresponding to

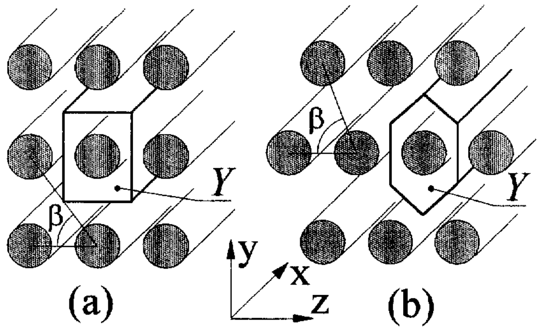

Fig. 1. Unidirectional fiber composite: (a) rectangular, and (b) hexagonal reinforcing arrays.

rectangular (Fig. 1a) and hexagonal (Fig. 1b) periodic reinforcing arrays. The characteristic parameter of the reinforcing array is the angle β, depicted in Fig. 1. The axes y and z lying in any cross-section of the composite are chosen as in Fig. 1.

Any unit cell of the real (i.e. heterogeneous) medium is associated with a material point of the equivalent homogenized medium. The stress and strain rate tensors at this point of the equivalent medium, denoted by $\underline{\underline{\Sigma}}$ and $\underline{\underline{D}}$, respectively, are defined as volume averages over the unit cell of their microscopic counterparts (denoted by $\underline{\underline{\sigma}}$ and $\underline{\underline{d}}$, respectively):

$$\underline{\underline{\Sigma}} = \frac{1}{|Y|}\int_Y \underline{\underline{\sigma}}(\underline{x})\,\mathrm{d}Y \equiv \langle\underline{\underline{\sigma}}\rangle; \quad \underline{\underline{D}} = \frac{1}{|Y|}\int_Y \underline{\underline{d}}(\underline{x})\,dY \equiv \langle\underline{\underline{d}}\rangle, \tag{1}$$

$\underline{x}$ being any point in the unit cell. By making use of Gauss's theorem, the latter of eqns (1) can also be expressed as a boundary integral,

$$\underline{\underline{D}} = \frac{1}{2|Y|}\int_{\partial Y} (\underline{v}\otimes\underline{n}+\underline{n}\otimes\underline{v})\,\mathrm{d}S, \tag{1'}$$

where $\underline{v}$ is the microscopic velocity field and $\underline{n}$ the outward normal to ∂Y; this form allows us to account for discontinuous velocity fields, with which a microscopic strain rate field can be associated anywhere but along possible discontinuity surfaces. Since further developments will be carried out in a "reference space", obtained through homothetic expansion of the physical space, for the sake of simplicity the cell volume can always be made equal to one ($|Y| = 1$).

The equivalent medium is supposed to be homogeneous and its mechanical properties are defined on the basis of the mechanical properties of the phases. In the following sections, the fundamental results of homogenization theory for periodic media applied to limit analysis will be recalled, with the aim of showing how the macroscopic strength of the composite can be predicted on the basis of the strength properties of the phases. For further details about this theoretical approach, readers are referred to Suquet (1983, 1987).

A general remark must be made about the theoretical approach employed here and the results it furnishes. The only data which are supposed to be available for the phases are their (convex) strength domains; no assumption is made regarding their constitutive law (i.e. no flow rule, either associated or not, is supposed to be known). This is quite an important point, since for many polymeric composites it seems quite unrealistic to postulate some kind of flow rule. This lack of information has obviously some drawbacks: in the following sections a domain in the space of the macroscopic stresses $\underline{\underline{\Sigma}}$ will be defined, which describes the overall strength properties of the composite; this domain has to be interpreted as the domain of "potentially safe" macroscopic stresses in the sense that stresses which do not fall within this domain cannot certainly be sustained by the composite, whereas no assurance exists that stresses which fall within it are sustainable. This domain takes the meaning of the set of actually admissible stresses, as in classic limit analysis, if an associated flow rule can be defined for the phases of the composite (see also Salençon, 1983, 1990).

2.1. *Lower bounds on the macroscopic uniaxial strength*

Let G_m, G_f and g_{int} denote the strength domains of the matrix, fiber, and fiber–matrix interface, respectively. The strength properties of the equivalent medium, i.e. the macroscopic strength of the composite, are characterized by a domain in the space of the macroscopic stresses which, if body forces are neglected, is defined as follows (Suquet, 1983):

$$\begin{aligned} G^{\mathrm{hom}} = \{\underline{\underline{\Sigma}} \mid \underline{\underline{\Sigma}} = \langle \underline{\underline{\sigma}} \rangle;\quad \underline{\underline{\sigma}} \cdot \underline{n} = \underline{T}\ \text{anti-periodic over}\ \partial Y; \\ \operatorname{div} \underline{\underline{\sigma}} = \underline{0}, [\![\underline{\underline{\sigma}}]\!] \cdot \underline{n}_S = \underline{0}\ \text{on}\ S; \\ \underline{\underline{\sigma}}(\underline{x}) \in G_m \forall \underline{x} \in Y_m;\quad \underline{\underline{\sigma}}(\underline{x}) \in G_f \forall \underline{x} \in Y_f;\quad \underline{T}(\underline{x}) \in g_{int} \forall \underline{x} \in S_{int}\}. \end{aligned} \tag{2}$$

Here, the symbol $[\![*]\!]$ stands for jump of the variable $*$ across any discontinuity surface S, with normal n_S. G^{hom} is a convex set if G_m, G_f and g_{int} are convex. Any microscopic stress field $\underline{\underline{\sigma}}(\underline{x})$ fulfilling all the conditions in eqn (2) is said to be *statically admissible* (s.a.).

Of course, G^{hom} would be completely known only if the entire class of the microscopic stress fields fulfilling the conditions in eqn (2) were explored. As is customary in limit analysis approaches, only subclasses of microscopic s.a. stress fields are considered, which allow the definition of lower bounds on G^{hom}.

Let $\underline{e}_x$ be the fiber axis unit vector. A piecewise constant s.a. microscopic stress field can be defined as follows:

$$\underline{\underline{\sigma}}(\underline{x}) = \underline{\underline{\sigma}}_m + \chi_f(\underline{x}) \sigma_f \underline{e}_x \otimes \underline{e}_x,$$

where $\otimes$ denotes tensor product and $\chi_f(\underline{x})$ is the characteristic function of the fiber (equals 1 if $\underline{x} \in Y_f$ and 0 otherwise). Based on this choice for $\underline{\underline{\sigma}}(\underline{x})$ and with some assumptions regarding the strength domains G_m and G_f recalled in the sequel, it was proved (see de Buhan and Taliercio, 1991; Taliercio, 1992) that a lower bound on G^{hom} is given by the domain

$$G_{0,\mathrm{int}} = G_0 \cap G_{\mathrm{int}} \tag{3}$$

with

$$G_0 = \{\underline{\underline{\Sigma}} \mid \underline{\underline{\Sigma}} = \underline{\underline{\sigma}}_m + \sigma \underline{e}_x \otimes \underline{e}_x;\quad \underline{\underline{\sigma}}_m \in G_m, -\bar{\sigma}^- \leqslant \sigma \leqslant \bar{\sigma}^+\} \tag{4}$$

and

$$G_{\mathrm{int}} = \{\underline{\underline{\Sigma}} \mid \underline{\underline{\Sigma}} \cdot \underline{n}(\underline{x}) = \underline{T}(\underline{x}) \in g_{\mathrm{int}} \forall \underline{x} \in S_{\mathrm{int}}\}. \tag{5}$$

In eqn (4), the bounds on the parameter σ are related to the uniaxial strengths of the fiber ($\sigma_f^\pm$) and the matrix ($\sigma_m^\pm$), and to the fiber volume fraction (η) as follows:

$$\bar{\sigma}^\pm = \eta(\sigma_f^\pm - \sigma_m^\pm).$$

Note that the lower bound $G_{0,\mathrm{int}}$ does not require the fiber strength domain G_f to be known, since only the uniaxial strengths of the fibers in tension and compression are involved in eqn (4). Also note that domain G_0 accounts for the strength properties of fibers and matrix, whereas G_{int} is dependent on the interface strength properties only. For composites with perfect bonding between fibers and matrix, g_{int} coincides with R^3 and the lower bound on G^{hom} reduces to G_0.

The inequality $G_{0,\text{int}} \subseteq G^{\text{hom}}$ was proved by Taliercio (1992) assuming that the matrix strength domain is not "less convex" than the fiber strength domain; in particular, this condition is fulfilled by homothetic domains (e.g. if both constituents are of the von Mises type). Going further into the mathematical details of this point would risk giving the impression of a highly speculative approach: here it suffices to say that the above assumption holds provided that the fibers are sufficiently stronger than the matrix, which will be implicitly assumed from here onwards. Recently, Angelillo and Aquilar (1992) proved the validity of this inequality for domains of any nature, provided that the strength domains of both the fiber and matrix are bounded. For further details, readers are referred to the quoted references.

Let us now focus attention on macroscopic uniaxial stress states acting in the (x, y) plane. Let Σ be the macroscopic stress and $\underline{e}_\vartheta$ be the unit vector defining its direction; the macroscopic stress tensor is then $\underline{\underline{\Sigma}} = \Sigma \underline{e}_\vartheta \otimes \underline{e}_\vartheta$. From the general definition of the lower bound $G_{0,\text{int}}$, eqn (3), it follows that a lower bound on the macroscopic uniaxial strength is given by

$$\Sigma^{\pm}_{0,\text{int}} = \min\{\Sigma^{\pm}_0, \Sigma^{\pm}_{\text{int}}\}$$

with

$$\Sigma^{\pm}_0(\vartheta) = \underset{\sigma}{\overset{+\sup}{\underset{\longrightarrow}{\overset{\longrightarrow}{-\inf}}}} \{\Sigma \mid \Sigma \underline{e}_\vartheta \otimes \underline{e}_\vartheta - \sigma \underline{e}_x \otimes \underline{e}_x \in G_{\text{m}}; \quad -\bar{\sigma}^- \leqslant \sigma \leqslant \bar{\sigma}^+\} \tag{6}$$

and

$$\Sigma^{\pm}_{\text{int}}(\vartheta) = \underset{x \in S_{\text{int}}}{\overset{+\sup}{\underset{\longrightarrow}{\overset{\longrightarrow}{-\inf}}}} \{\Sigma \mid \Sigma \underline{e}_\vartheta \otimes \underline{e}_\vartheta \cdot \underline{n}(\underline{x}) \in g_{\text{int}}\}. \tag{7}$$

It is interesting to understand the meaning of "weakening interface" as far as the lower bound is concerned, that is, when the lower bound $\Sigma^{\pm}_{0,\text{int}}$ computed accounting for an interface criterion is more restrictive than its counterpart $\Sigma^{\pm}_0$ based on perfect bonding. This may not be obvious, unless the strength criteria of matrix and interface are of the same kind. Actually, based on eqn (3) it turns out that the composite strength properties are weakened by an interface of finite strength if $G_0 \cap G_{\text{int}} \subset G_0$, where perfect bonding is assumed in computing G_0. In the uniaxial case, this amounts to $\Sigma^{\pm}_{0,\text{int}} < \Sigma^{\pm}_0$. If the two domains G_0 and G_{int} intersect, the interface weakens the composite when subjected to particular stresses. Finally, if $G_0 \subset G_0 \cap G_{\text{int}}$, the interface does not weaken the composite and perfect bonding can be assumed.

2.2. *Upper bounds on the macroscopic uniaxial strength*

Since G^{hom} is convex, it can be dually defined through its own *support function*, $\pi^{\text{hom}}(\underline{\underline{D}})$, $\underline{\underline{D}}$ being any element of R^6 (see e.g Tyrrell Rockafellar, 1970):

$$G^{\text{hom}} = \{\underline{\underline{\Sigma}} \mid \underline{\underline{\Sigma}} : \underline{\underline{D}} \leqslant \pi^{\text{hom}}(\underline{\underline{D}}), \forall \underline{\underline{D}} \in R^6\}. \tag{8}$$

The definition of π^{hom} is also due to Suquet (1983) and reads

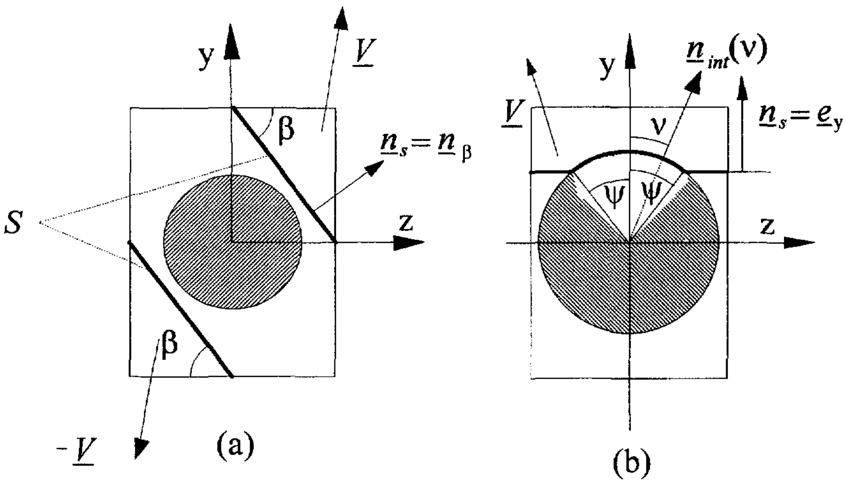

Fig. 2. Failure mechanisms of a rectangular unit cell: (a) planes crossing the matrix, and (b) surface involving the fiber–matrix interface.

$$\pi^{\text{hom}}(\underline{\underline{D}}) = \inf_{\underline{v}} \left\{ P_y(\underline{v}) \mid \underline{\underline{D}} = \frac{1}{2} \int_{\partial Y} (\underline{v} \otimes \underline{n} + \underline{n} \otimes \underline{v}) \, \mathrm{d}S \right\}, \tag{9}$$

where $\underline{v}(\underline{x})$ is any vector field of the form

$$\underline{v}(\underline{x}) = \underline{\underline{D}} \cdot \underline{x} + \underline{u}(\underline{x}), \tag{10}$$

$\underline{u}(\underline{x})$ being any Y-periodic piecewise differentiable vector field, and where

$$P_y(\underline{v}) = \int_Y \pi(\underline{\underline{d}}) \, \mathrm{d}Y + \int_S \pi(\underline{V}; \underline{n}) \, \mathrm{d}S.$$

Here π indicates the support function of the strength domain of the material forming the unit cell at any point; $\underline{\underline{d}}$ is the symmetric part of the gradient of $\underline{v}$ at any point where $\underline{v}$ is differentiable; finally, S is any discontinuity surface for $\underline{v}$ in Y, $\underline{n}$ is the normal to S at any point, $\underline{V} = [\![\underline{v}]\!]$ across S and $\pi(\underline{V}; \underline{n}) = 1/2\pi(\underline{V} \otimes \underline{n} + \underline{n} \otimes \underline{V})$.

It is worth noting that, if $\underline{v}$ is interpreted as the microscopic velocity field, $\underline{\underline{d}}$ turns out to be the corresponding microscopic strain rate field and $\underline{\underline{D}}$ is the macroscopic strain rate of the unit cell. Since $\underline{v}$ is in general discontinuous, in eqn (9) use was made of eqn (1′) in computing $\underline{\underline{D}}$. Any velocity field that can be expressed in the form (10) is said to be *kinematically admissible* (k.a.); P_y can be interpreted as power dissipated in any k.a. failure mechanism for the unit cell; $\pi^{\text{hom}}(\underline{\underline{D}})$ takes the mechanical meaning of power dissipated in the real failure mechanism of the cell characterized by a prescribed $\underline{\underline{D}}$. For further details, see also de Buhan (1986).

G^{hom} would be completely known through its dual definition, eqn (8), if the entire class of the microscopic k.a. failure mechanisms for the cell were explored. If only subclasses of microscopic k.a. velocity fields are considered, upper bounds on G^{hom} are obtained. In particular, if a failure mechanism with uniform strain rate ($\underline{\underline{d}}(\underline{x}) = \underline{\underline{D}} = \text{const.}$) is considered, the bound

$$\pi^{\text{hom}}(\underline{\underline{D}}) \leqslant \pi_{\text{m}}(\underline{\underline{D}})(1-\eta) + \pi_{\text{f}}(\underline{\underline{D}})\eta$$

is obtained, π_{m} and π_{f} being the support functions of G_{m} and G_{f}, respectively. If a failure mechanism characterized by relative movement of parts of the cell supposed to be rigid is considered, denoting by $\underline{V}$ the relative velocity of the rigid parts, the following bounds are obtained:

- if the failure surface S is a plane entirely passing through the matrix, denoting by $\underline{n}_S$ the normal to the plane (see Fig. 2a), one gets

$$\pi^{\text{hom}}(\underline{\underline{D}}) \leqslant |S|\,\pi_{\text{m}}(\underline{V};\underline{n}_S),$$

where $|S|$ is the dimension of S in the (y, z) plane;

- if the failure surface S consists of a flat part cutting through the matrix, S_{m}, with normal $\underline{n}_S$, and a curved part of the fiber–matrix interface (see Fig. 2b), one gets

$$\pi^{\text{hom}}(\underline{\underline{D}}) \leqslant \min_{\psi} |S_{\text{m}}(\psi)|\,\pi_{\text{m}}(\underline{V};\underline{n}_S) + \int_{-\psi}^{+\psi} \pi_{\text{int}}(\underline{V};\underline{n}_{\text{int}}(v))R\,\text{d}v; \tag{11}$$

here π_{int} is the support function of the interface strength domain, ψ is half of the angle subtended by the curved part of the failure surface and R is the fiber radius.

In both cases, the corresponding macroscopic strain rate is $\underline{\underline{D}} = 1/2(\underline{V}\otimes \underline{n}_S+\underline{n}_S\otimes \underline{V})$.

Note that the failure surfaces must be compatible with the periodicity of the medium; this means that $\underline{n}_S$ cannot be arbitrary, but it must either coincide with one of the unit vectors of the axes lying in the plane perpendicular to the fiber axis ($\underline{n}_S = \underline{e}_y$ or $\underline{e}_z$) or with $\underline{n}_\beta = \{0, \pm c_\beta, \pm s_\beta\}$, where $c_\beta = \cos\beta$ and $s_\beta = \sin\beta$.

It is understood that the velocity fields involved in the above expressions must give finite values for the dissipated power if significant upper bounds are to be obtained. For instance, if the phases comply with the von Mises criterion, only the mechanisms characterized by slipping along failure planes should be accounted for. This point will be more thoroughly discussed in Section 3.

On the basis of the above failure mechanisms, it is possible, in particular, to formulate upper bounds on the strength of the material subjected to any uniaxial macroscopic stress acting in the (x, y) plane. These upper bounds, $\Sigma_{\text{I}}^{\pm}(\vartheta)$, can be expressed as

$$\Sigma_{\text{I}}^{\pm} = \min\{\Sigma_h^{\pm}, \Sigma_y^{\pm}, \Sigma_\beta^{\pm}\}.$$

Here $\Sigma_h^{\pm}$ is the upper bound given by the mechanism characterized by constant strain rate:

$$\Sigma_h^{\pm}(\vartheta) = \inf_{\underline{\underline{D}}}\left\{\frac{\pi_{\text{m}}(\underline{\underline{D}})(1-\eta)+\pi_{\text{f}}(\underline{\underline{D}})\eta}{\pm D_{\vartheta\vartheta}}\right\},$$

where $D_{\vartheta\vartheta}$ is the normal component of the macroscopic strain rate along $\underline{e}_\vartheta$. Assuming that the strength domains of the matrix and fiber fulfil the same requirements as in Section 2.1, where the lower bound $G_{0,\text{int}} \subseteq G^{\text{hom}}$ was introduced, this upper bound turns out to be isotropic and coincides with the result of the so-called "rule of mixtures":

$$\Sigma_h^{\pm} = \sigma_{\text{m}}^{\pm}(1-\eta)+\sigma_{\text{f}}^{\pm}\eta. \tag{12}$$

This value coincides with the one given by the lower bound in uniaxial tension/compression at $\vartheta = 0°$ (see e.g. Taliercio, 1992): this means that the macroscopic uniaxial strength along the fibers is exactly determined and is given by eqn (12).

In the case of failure planes involving the matrix only, the other upper bounds are:

$$\Sigma_y^{\pm}(\vartheta) = \inf_{\underline{V}}\frac{\pi_{\text{m}}(\underline{V};\underline{e}_y)}{\pm(V_x c_\vartheta + V_y s_\vartheta)s_\vartheta}; \tag{13}$$

$$\Sigma_\beta^{\pm}(\vartheta) = \inf_{\underline{V}}\frac{\pi_{\text{m}}(\underline{V};\underline{n}_\beta)}{\pm(V_x c_\vartheta + V_y s_\vartheta)s_\vartheta c_\beta}, \tag{14}$$

where $s_\vartheta = \sin\vartheta$ and $c_\vartheta = \cos\vartheta$. Mechanisms with failure planes perpendicular to $\underline{e}_z$ are not accounted for here, since they do not yield significant upper bounds for the considered uniaxial macroscopic stress.

Of course, failure planes not intersecting any fiber can be drawn in the composite only if the reinforcing array fulfils certain conditions on η and β that can be obtained based on purely geometrical considerations:

- in the case of *rectangular* reinforcing arrays, if $\eta \leqslant \pi/8 \sin 2\beta$ failure planes perpendicular to $\underline{n}_\beta$ are possible, whereas failure planes perpendicular to $\underline{e}_y$ can be drawn for any η, β;
- in the case of *hexagonal* reinforcing arrays, the conditions that must be fulfilled in order that failure planes perpendicular to $\underline{n}_\beta$ or to $\underline{e}_y$ do not intersect any fiber are, respectively, $\eta \leqslant \pi/4 \sin 2\beta$ and $\eta \leqslant \pi/8\, \mathrm{tg}\beta$.

If the previous geometrical conditions are not satisfied, the failure surfaces are partly made up of the fiber–matrix interface and more involved expressions for the upper bounds are obtained, as will be shown in Section 3.

It must be acknowledged that failure mechanisms of the kind described above were used by Majumdar and McLaughlin (1975) to derive upper bounds on the strength of unidirectional composites with matrices obeying the von Mises limit condition. Even though these authors did not make use of homogenization concepts, they were aware of the necessity of defining failure mechanisms characterized by the same periodicity as the composite material, in order to bound its macroscopic strength from above. Actually, their results can be applied to metal–matrix composites (provided that the fiber content is not too high, so that failure planes cutting through the matrix only can be drawn), for which it is reasonable to make use of the von Mises condition; they are not of much interest for polymeric–matrix composites.

2.3. *Remarks on the main characteristics of the model*

The approach employed here for the formulation of approximate macroscopic failure criteria is of the micromechanical type. This means that the strength domains it provides are not merely an interpolation of experimental points, as is the case with the majority of the existing criteria for composites (e.g. the Tsai–Wu criterion), but contain information regarding the way in which the composite reaches a limit state because of the failure of one or more of its phases.

Towards the end of illustrating more clearly this point, it seems advisable to anticipate some of the results that will be derived and more thoroughly discussed in the following sections. In Fig. 3 the bounds on the uniaxial compressive strength of a composite (Σ^-) obtained through application of the present model are shown. Square symbols correspond to the lower bound ($\Sigma^-_{0,\mathrm{int}}$), whereas triangles correspond to the upper bound (Σ^-_{I}). In this example, the matrix is presumed to comply with a Drucker–Prager type strength criterion,

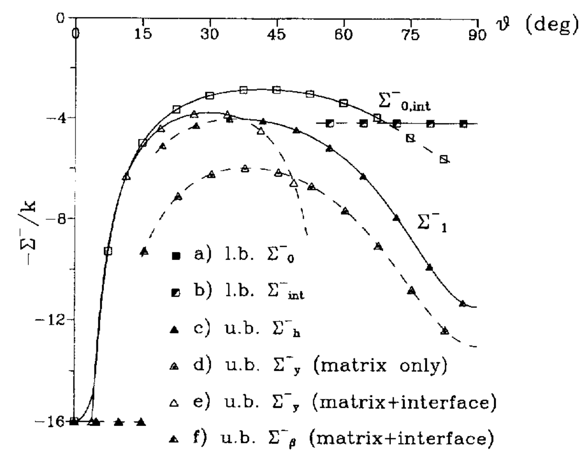

Fig. 3. Bounds on the uniaxial compressive strength vs fiber orientation (ϑ): meaning of the different branches forming the lower bound ($\Sigma^-_{0,\mathrm{int}}$) and the upper bound ($\Sigma^-_{\mathrm{I}}$) curves.

whereas the strength of the interface is modelled using the Coulomb criterion. These criteria are amongst the ones used in the following Section 3, but different choices can be made and incorporated in the model. The curves in Fig. 3 were plotted according to the equations reported in Section 3; details are given below. The strength values in Fig. 3 were normalized by a parameter k, related to the shear strength of the matrix. In the example shown, a square reinforcing array ($\beta = 45°$) and a volume fraction $\eta = 0.5$ were considered. The matrix compressive-to-tensile strength ratio was taken equal to 2, the interface pure shear strength to $1.2k$ and the interface friction angle to 30°. The fiber strength was assumed to be about 10 times greater than the matrix strength in compression. Since the aim of Fig. 3 is mainly to demonstrate the model capabilities, these values are not necessarily representative of the strength of real composites and were just selected for the sake of illustration.

Referring to the lower bound curve, it is possible to locate two branches. The first one (curve a) does not account for the interface limited strength and corresponds to the bound Σ_0^- of Section 2.1 [which, for Drucker–Prager matrices, takes the form of eqn (24)]. The second one (curve b) corresponds to the attainment of a limit condition at some point of the interface, i.e. to the bound Σ_{int}^- of Section 2.1 [which, for Coulomb-type interfaces, takes the form of eqn (35)].

As for the upper bound, several branches can be located along the relevant curve. The horizontal line starting from $\vartheta = 0$ (curve c) corresponds to the uniform strain rate failure mechanism [i.e. to Σ_h^-, eqn (12)]. The intermediate branch (curve e) is generated by a mechanism with failure surface consisting of part of the fiber–matrix interface and of a plane cutting the matrix perpendicular to $\underline{e}_y$ [see eqn (37)]. Also note the presence in the plot of a curve (d) entirely lying below the upper bound, which was obtained using a failure mechanism with plane perpendicular to $\underline{e}_y$ passing through the matrix only; it corresponds to the bound Σ_y^- of Section 2.2 which, for Coulomb-type matrices, takes the form of eqn (26) (with $c_\beta = 1$). Since the interface was supposed to be "weaker" than the matrix, the power required to fail the composite is less if the failure surface passes through part of the fiber–matrix interface, rather than through the matrix only: this is why curve d overestimates the strength of the considered composite. Note that both types of mechanisms fulfil the periodicity requirements (see also Fig. 9a,b in Section 3). At angles greater than about 35° the best upper bound (curve f) is given by a mechanism characterized by a failure surface formed by part of a plane perpendicular to $\underline{n}_\beta$ and part of the fiber–matrix interface: the interface is necessarily involved in this kind of mechanism because of the geometry of the reinforcing array, so that the bound Σ_β^- of Section 2.2 has to be modified [solving eqn (40) of Section 3.5].

3. INFLUENCE OF MATRIX AND INTERFACE UPON THE MACROSCOPIC UNIAXIAL STRENGTH

In this section, approximate expressions for the relationship between uniaxial macroscopic strength and fiber orientation will be derived for composites with isotropic phases of different nature. Initially, the influence of the interface strength properties upon the strength of the composite will be neglected, i.e. composites with perfect bonding between fibers and matrix will be considered: this restriction will be removed in the last part of this section (Section 3.5).

The problem arises of determining which strength criteria allow a realistic simulation of the ultimate properties of the phases of widest use in the fabrication of fiber-reinforced composites. Note first of all that the material of which the fibers are made up is involved in the bounds formulated in Section 2 only through its uniaxial tensile/compressive strength [see eqns (6) and (12)]: in other words, it is not necessary to know the entire fiber strength criterion.

As regards the matrix strength criterion, attention will be focused here on polymeric-matrix composites. For polymeric matrices, the experimental results available in the literature show two main features, namely that the matrix strength is influenced by a hydrostatic pressure and that the uniaxial strengths of the matrix in tension and compression are different. Referring for instance to the results reported by Hull (1981) on different polymers submitted to biaxial stress to failure, the shape of the yield locus seems to suggest that a Drucker–Prager or parabolic-type strength criterion would be adequate for their description. On the other hand, the failure data obtained by Kawabata (1982) on epoxy resin samples tested under biaxial stress are fairly well interpolated by a Coulomb-type strength criterion—as proposed by the author himself.

In light of these observations, in the following sections the general bounds formulated in Section 2 will be specialized to matrix strength criteria of the three kinds quoted above. The upper bound on the macroscopic uniaxial strength expressed by eqn (12), $\Sigma_h^{\pm}$, is not influenced by the choice of the matrix strength criterion, so it will be left out from further considerations.

For the sake of brevity, the subscript m featuring any quantity referred to the matrix will be dropped in the sequel. Use will be made of the symbols J_1 and J'_2, that will denote the first invariant of the stress ($\underline{\underline{\sigma}}$) and the second invariant of the deviatoric stress ($\underline{\underline{s}}$), respectively:

$$J_1 = \mathrm{tr}\underline{\underline{\sigma}}; \quad J'_2 = 1/2\,\mathrm{tr}\underline{\underline{s}}^2 = 1/2\,(\mathrm{tr}\underline{\underline{\sigma}}^2 - 1/3\,\mathrm{tr}^2\underline{\underline{\sigma}}).$$

Similarly, the symbols I_1 and I'_2 will denote the corresponding quantities in terms of strain rates, $\underline{\underline{d}}$:

$$I_1 = \mathrm{tr}\underline{\underline{d}}; \quad I'_2 = 1/2\,(\mathrm{tr}\underline{\underline{d}}^2 - 1/3\,\mathrm{tr}^2\,\underline{\underline{d}}).$$

3.1. *Matrix obeying a parabolic-type strength criterion*

Consider first the case of a matrix made up of a material whose strength properties are described through a parabolic-type criterion:

$$G = \{\underline{\underline{\sigma}} \mid J'_2 + 2/3aJ_1 \leqslant k^2_{(P)}\}, \tag{15}$$

where a, $k_{(P)}$ are strength parameters related to the uniaxial tensile/compressive strength values by the relationships

$$a = \frac{\sigma^- - \sigma^+}{2}; \quad k_{(P)} = \sqrt{\frac{\sigma^-\sigma^+}{3}};$$

$k_{(P)}$ is the pure shear strength of the matrix. This criterion reduces to the von Mises criterion for materials having equal uniaxial strength in tension and compression. The use of this criterion was suggested by Hull (1981) for thermosetting resins; a similar criterion, formulated in the strain space rather than in the stress space, was also proposed by Christensen (1988, 1990) to model the "fiber–matrix interaction" in the failure of composites with epoxy matrices.

The support function of the domain defined by eqn (15) reads

$$\pi(\underline{\underline{d}}) = \begin{cases} 2a\dfrac{I'_2}{I_1} + \dfrac{k^2_{(P)}}{2a}I_1 & \text{if } I_1 \geqslant 0 \\ +\infty & \text{if } I_1 < 0 \end{cases}. \tag{16}$$

This expression will be explicitly derived in Appendix 1.

In the case of velocity fields characterized by a jump $\underline{V}$ across a discontinuity surface with normal $\underline{n}$, eqn (16) specializes to

$$\pi(\underline{V},\underline{n}) = \begin{cases} \dfrac{a}{2}\dfrac{|\underline{V}|^2}{\underline{V}\cdot\underline{n}} + \dfrac{1}{2}\left(\dfrac{a}{3} + \dfrac{k_{(P)}^2}{a}\right)\underline{V}\cdot\underline{n} & \text{if } \underline{V}\cdot\underline{n} \geqslant 0 \\ +\infty & \text{if } \underline{V}\cdot\underline{n} < 0 \end{cases} . \tag{17}$$

Equation (17) shows that any failure mechanism characterized by separation of a unit cell by a surface cutting through a parabolic-type matrix is admissible.

Suppose that the composite is subjected to a macroscopic uniaxial stress Σ, acting in the (x, y) plane at an angle ϑ to the fibers. If the static approach shown in Section 2.1 is considered, the stress in the matrix is $(\underline{\sigma} =)\underline{\sigma}_m = \Sigma \underline{e}_\vartheta \otimes \underline{e}_\vartheta - \sigma \underline{e}_x \otimes \underline{e}_x$. By specializing the expression of the stress invariants in eqn (15) to the uniaxial case, i.e.

$$J_1 = \Sigma - \sigma, \quad 3J_2' = \Sigma^2 - 2\Sigma\sigma(1 - 3/2s_\vartheta^2) + \sigma^2$$

and substituting in eqn (6), a constrained maximization problem is obtained which yields a lower bound on the macroscopic uniaxial strength of composites with parabolic-type matrix and perfect bonding at the fiber–matrix interface:

$$\Sigma_0^{\pm}(\vartheta) = \begin{cases} +\sup \\ -\inf_{\sigma} \end{cases} \{\Sigma | \Sigma^2 - 2[\sigma(1 - 3/2s_\vartheta^2) - a]\Sigma - 2a\sigma + \sigma^2 - 3k_{(P)}^2 \leqslant 0; \quad -\bar{\sigma}^- \leqslant \sigma \leqslant \bar{\sigma}^+\}.$$

Solving this problem for the lower bound on the compressive strength, the following expressions are obtained:

$$\Sigma_0^-(\vartheta) = \begin{cases} \bar{\sigma}^-(1 - 3/2s_\vartheta^2) + a + \sqrt{a^2 + 3k_{(P)}^2 - 3s_\vartheta^2\bar{\sigma}^-[a + \bar{\sigma}^-(1 - 3/4s_\vartheta^2)]} & \text{if } \tilde{\sigma} \geqslant \bar{\sigma}^- \\ \dfrac{a}{2(1 - 3/4s_\vartheta^2)} + \dfrac{\sqrt{a^2 + 3k_{(P)}^2(1 - 3/4s_\vartheta^2)}}{\sqrt{3}s_\vartheta(1 - 3/4s_\vartheta^2)} & \text{if } -\bar{\sigma}^+ < \tilde{\sigma} < \bar{\sigma}^- \\ -\bar{\sigma}^+(1 - 3/2s_\vartheta^2) + a + \sqrt{a^2 + 3k_{(P)}^2 - 3s_\vartheta^2\bar{\sigma}^+[a + \bar{\sigma}^+(1 - 3/4s_\vartheta^2)]} & \text{if } \tilde{\sigma} \leqslant -\bar{\sigma}^+ \end{cases}$$

(18a, b, c)

where

$$\tilde{\sigma} = -\frac{a}{2(1 - 3/4s_\vartheta^2)} + \frac{(1 - 3/2s_\vartheta^2)\sqrt{a^2 + 3k_{(P)}^2(1 - 3/4s_\vartheta^2)}}{\sqrt{3}s_\vartheta(1 - 3/4s_\vartheta^2)}. \tag{19}$$

The lower bound on the uniaxial macroscopic tensile strength, $\Sigma_0^+(\vartheta)$, can be derived from eqns (18a,b) and (19), by inverting the roles of $\bar{\sigma}^+$ and $\bar{\sigma}^-$ and by changing a into $-a$. Eqn (18c) can be left out of consideration, since under macroscopic uniaxial tension the bound $\tilde{\sigma} \leqslant -\bar{\sigma}^-$ is never exceeded.

In the case of parabolic-type matrices, the upper bound expressed by eqn (14) specializes to

$$\Sigma_\beta^{\pm}(\vartheta) = \frac{\mp a + \sqrt{a^2 - (1 - s_\vartheta^2 c_\beta^2)(4/3a^2 + k_{(P)}^2)/s_\vartheta^2 c_\beta^2}}{1 - s_\vartheta^2 c_\beta^2}. \tag{20}$$

Similarly, the upper bound yielded by eqn (13), $\Sigma_y^{\pm}(\vartheta)$, can be obtained by the above eqn (20) by putting $\beta = 0$ (i.e. $c_\beta = 1$).

A parametric study was carried out in order to investigate the influence of the matrix strength parameters on the bounds for the macroscopic strength of the composite. The

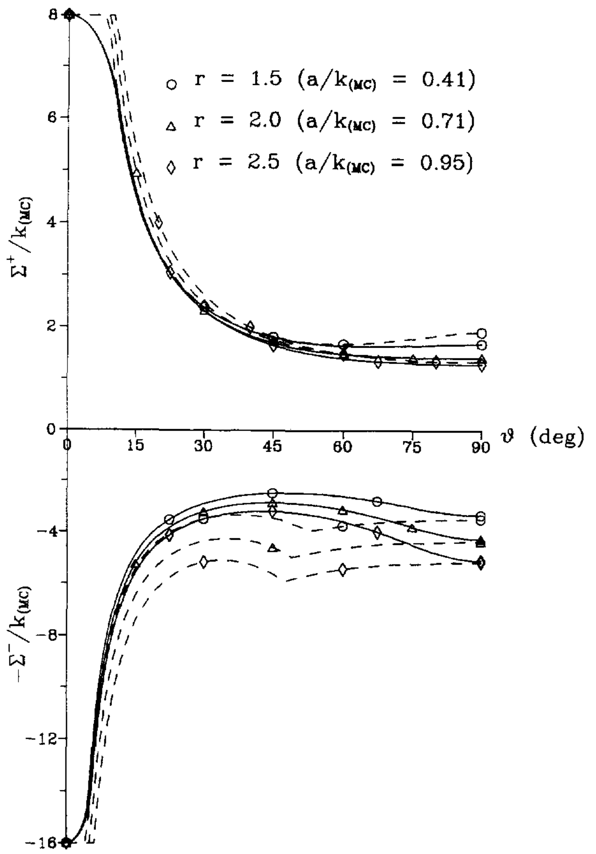

Fig. 4. Bounds on the uniaxial strength of composites with matrix obeying a parabolic-type criterion for different ratios r between uniaxial tensile and compressive matrix strength. Hexagonal reinforcing array, $\eta = 0.65$, $\beta = 60°$. Solid lines are lower bounds, dashed lines are upper bounds.

results are presented in Fig. 4, where the macroscopic tensile/compressive strength of a composite with parabolic-type matrix is plotted vs the fiber orientation ϑ for different values of the ratio $r = \sigma^-/\sigma^+$; the macroscopic strength is normalized by a strength parameter $k_{(MC)} = (\sqrt{3}/2)k_{(P)}$. As a rule, the upper bounds are relatively close to the relevant lower bounds for any orientation ϑ; this is particularly true for the bounds in tension. In the range of r values explored, amongst the mechanisms characterized by failure planes considered in Section 2.2, the mechanism that yields the best (i.e. most restrictive) upper bound on the tensile strength at any orientation ϑ and on the compressive strength at $\vartheta \leqslant 45°$ turned out to be the one with failure plane perpendicular to the applied tension (corresponding to Σ_y^+). At angles ϑ greater than about 45°, the best upper bound on the compressive strength is given by the failure mechanism with oblique failure plane (corresponding to Σ_β^-). In particular, the strength of the composite subjected to uniaxial stress transverse to the fibers (i.e. at $\vartheta = 90°$) is located between the two bounds

$$\Sigma_0^+(90°) = 2[-a+\sqrt{4/3a^2+k_{(P)}^2}] \quad \text{and} \quad \Sigma_y^+(90°) = 1/2(1/3a+k_{(P)}^2/a)$$

in tension and between

$$\Sigma_0^-(90°) = 2[a+\sqrt{4/3a^2+k_{(P)}^2}] \quad \text{and} \quad \Sigma_\beta^-(90°) = \frac{a+\sqrt{a^2-(4/3a^2+k_{(P)}^2)\mathrm{tg}^2\beta}}{s_\beta^2}$$

in compression. It was implicitly assumed that under macroscopic compression the bound on the tensile strength of the fibers is not active [i.e. use was made of eqn (18a) in computing

$\Sigma_0^-(90^\circ)$ and not of eqn (18c)], which is the case if the difference between the tensile and compressive strengths of the matrix is not excessive.

3.2. *Matrix obeying the Drucker–Prager strength criterion*

Consider now the case where the matrix strength properties are described by a Drucker–Prager type criterion:

$$G = \{\underline{\underline{\sigma}} \mid \sqrt{J_2'} + \alpha J_1 \leqslant k_{(\mathrm{DP})}\}. \tag{21}$$

Again, $k_{(\mathrm{DP})}$ is the pure shear strength of the matrix, whereas α is a nondimensional parameter accounting for the unequal behaviour of the material in tension and compression; the equations relating these parameters to the uniaxial strengths of the matrix are

$$\alpha = \frac{1}{\sqrt{3}}\frac{\sigma^- - \sigma^+}{\sigma^- + \sigma^+}; \quad k_{(\mathrm{DP})} = \frac{2}{\sqrt{3}}\frac{\sigma^-\sigma^+}{\sigma^- + \sigma^+}.$$

Values of α greater than $1/\sqrt{12}$ are not physically admissible, which means that this criterion does not allow the description of the limit behaviour of materials with the compressive-to-tensile strength ratio greater than 3.

The support function of the Drucker–Prager strength domain reads (see e.g. Salençon, 1983):

$$\pi(\mathrm{d}) = \begin{cases} \dfrac{k_{(\mathrm{DP})}}{3\alpha} I_1 & \text{if } I_1 \geqslant 6\alpha\sqrt{I_2'} \\ +\infty & \text{if } I_1 < 6\alpha\sqrt{I_2'} \end{cases}, \tag{22}$$

whereas, in the case of discontinuous velocity fields, eqn (22) specializes to

$$\pi(\underline{V}, \underline{n}) = \begin{cases} \dfrac{k_{(\mathrm{DP})}}{3\alpha} \underline{V}\cdot\underline{n} & \text{if } \dfrac{\underline{V}\cdot\underline{n}}{|\underline{V}|} \geqslant \dfrac{3\alpha}{\sqrt{1-3\alpha^2}} \\ +\infty & \text{if } \dfrac{\underline{V}\cdot\underline{n}}{|\underline{V}|} < \dfrac{3\alpha}{\sqrt{1-3\alpha^2}} \end{cases}. \tag{23}$$

Equation (23) means that the failure mechanisms characterized by separation of a unit cell are admissible if the angle between the jump in velocity, $\underline{V}$, and the normal $\underline{n}$ to the discontinuity surface does not exceed a certain value ($\arccos 3\alpha/\sqrt{1-3\alpha^2}$) at any point of the surface.

The results obtained through application of the present model to composites with Drucker–Prager matrices were already presented (see Taliercio, 1992) and are briefly reported here for the sake of completeness.

The lower bound on the compressive macroscopic strength is expressed by the equations:

$$\Sigma_0^-(\vartheta) = \begin{cases} \bar{\sigma}^- + \dfrac{\sqrt{3}}{1-3\alpha^2}\sqrt{(3/2\alpha\bar{\sigma}^- s_\vartheta^2 + k_{(\mathrm{DP})})^2 - (\bar{\sigma}^-)^2(1-3/4s_\vartheta^2)(1-3\alpha^2)s_\vartheta^2} \\ \qquad + 3\dfrac{-1/2\bar{\sigma}^- s_\vartheta^2 + \alpha k_{(\mathrm{DP})}}{1-3\alpha^2} & \text{if } \bar{\sigma} \geqslant \bar{\sigma}^- \\ \left[\dfrac{3/2\alpha}{1-3\alpha^2-3/4s_\vartheta^2} + \dfrac{\sqrt{(1-3/4s_\vartheta^2)(1-3\alpha^2)}}{(1-3\alpha^2-3/4s_\vartheta^2)s_\vartheta}\right] k_{(\mathrm{DP})} & \text{if } -\bar{\sigma}^+ \leqslant \sigma \leqslant \bar{\sigma}^- \\ -\bar{\sigma}^+ + \dfrac{\sqrt{3}}{1-3\alpha^2}\sqrt{(3/2\alpha\bar{\sigma}^+ s_\vartheta^2 + k_{(\mathrm{DP})})^2 - (\bar{\sigma}^+)^2(1-3/4s_\vartheta^2)(1-3\alpha^2)s_\vartheta^2} \\ \qquad + 3\dfrac{1/2\bar{\sigma}^+ s_\vartheta^2 + \alpha k_{(\mathrm{DP})}}{1-3\alpha^2} & \text{if } \sigma \leqslant -\bar{\sigma}^+ \end{cases}$$

(24a, b, c)

where

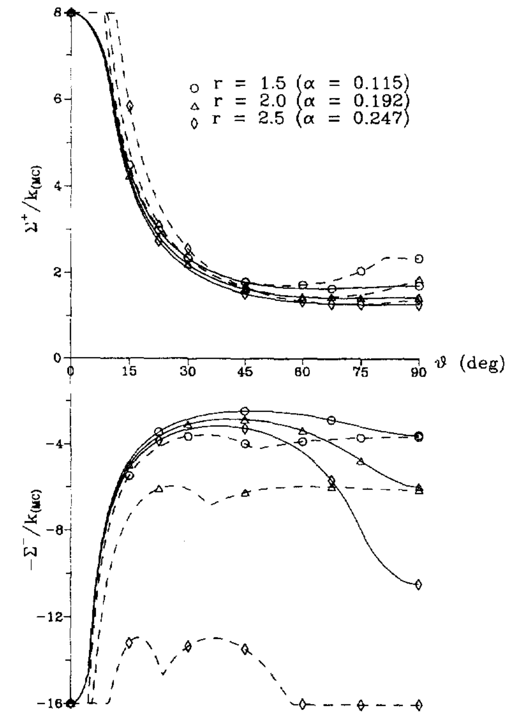

Fig. 5. Bounds on the uniaxial strength of composites with matrix obeying the Drucker–Prager criterion for different ratios r between uniaxial tensile and compressive matrix strength. Hexagonal reinforcing array, $\eta = 0.65$, $\beta = 60°$. Solid lines are lower bounds, dashed lines are upper bounds.

$$\tilde{\sigma} = \left(\frac{3/2\alpha}{1-3\alpha^2-3/4s_\vartheta^2} - \frac{\sqrt{1-3/4s_\vartheta^2}}{\sqrt{1-3\alpha^2}s_\vartheta} \frac{1-3\alpha^2-3/2s_\vartheta^2}{1-3\alpha^2-3/4s_\vartheta^2} \right) k_{(DP)}. \tag{25}$$

The lower bound on the uniaxial macroscopic tensile strength, $\Sigma_0^+(\vartheta)$, can be obtained from eqns (24a,b) and (25), by inverting the roles of $\bar{\sigma}^+$ and $\bar{\sigma}^-$ and by changing α into $-\alpha$; eqn (24c) can be left out of consideration, since under macroscopic uniaxial tension the bound $\tilde{\sigma} \leqslant -\bar{\sigma}^-$ is never exceeded.

The upper bound generated by mechanisms with oblique failure planes, $\Sigma_\beta^\pm$, is given by

$$\Sigma_\beta^\pm = \frac{k_{(DP)}}{(\sqrt{1-12\alpha^2}\sqrt{1-s_\vartheta^2 c_\beta^2} \pm 3\alpha s_\vartheta c_\beta) s_\vartheta c_\beta}; \tag{26}$$

similarly, the upper bound generated by the mechanism with failure plane normal to $\underline{e}_y$ can be obtained by putting $c_\beta = 1$ in eqn (26).

The influence of Drucker–Prager parameters on the bounds for the macroscopic strength of the composite was investigated, leading to the results shown in Fig. 5. Again, the macroscopic strength is plotted vs the orientation of the applied stress to the fibers, for different ratios r between uniaxial compressive and tensile strength of the matrix; the bounds on the macroscopic strength are normalized by a strength parameter $k_{(MC)} = k_{(DP)}/(2\sqrt{1/3-\alpha^2})$. Similar to the previous case of a parabolic-type matrix, the gap between the bounds on the tensile strength is relatively small. Much wider is the range between the bounds on the compressive strength of the composite; this is particularly true if the matrix behaviour in tension and compression is much different (i.e. as α increases). An extensive discussion on the reasons for this is found in the first part of Section 3.4.

In light of this parametric study, the failure mechanism yielding the best upper bound on the macroscopic uniaxial strength turns out to be conditional upon the uniaxial strengths of the matrix and fibers. Same arguments hold with reference to the lower bounds; thus, it is rather complicated to give general equations for the bounds on the macroscopic strength of a composite with Drucker–Prager matrix subjected, for instance, to tension or compression transverse to the fibers.

3.3. *Matrix obeying the Mohr–Coulomb strength criterion*

The last type of matrix considered is one complying with the Mohr–Coulomb strength criterion, which can be expressed as

$$G = \{\underline{\underline{\sigma}} \,|\, |\tau(\underline{n})| \leqslant k_{(\mathrm{MC})} - \sigma(\underline{n})\mathrm{tg}\varphi, \quad \forall \underline{n}; \sigma = \underline{n}\cdot\underline{\underline{\sigma}}\cdot\underline{n}, \quad |\tau|^2 = \underline{n}\cdot\underline{\underline{\sigma}}\cdot\underline{\underline{\sigma}}\cdot\underline{n} - \sigma^2\} \tag{27}$$

or, equivalently, as

$$G = \{\underline{\underline{\sigma}} \,|\, \sigma_{\max}\mathrm{tg}(45^\circ + \varphi/2) - \sigma_{\min}\mathrm{tg}(45^\circ - \varphi/2) \leqslant k_{(\mathrm{MC})}\}, \tag{27'}$$

where $\sigma_{\max}$, $\sigma_{\min}$ are the maximum and minimum principal stresses in the matrix. $k_{(\mathrm{MC})}$ and φ are the pure shear strength (cohesion) and the internal angle of friction of the matrix; these parameters are related to the uniaxial strengths of the matrix by the relationships

$$\sin\varphi = \frac{\sigma^- - \sigma^+}{\sigma^- + \sigma^+}; \quad k_{(\mathrm{MC})} = 1/2\sqrt{\sigma^-\sigma^+}.$$

The support function of the Coulomb strength domain is (see e.g. Salençon, 1983):

$$\pi(\underline{\underline{d}}) = \begin{cases} HI_1 & \text{if } I_1 \geqslant |d_{\mathrm{I}}| + |d_{\mathrm{II}}| + |d_{\mathrm{III}}| \\ +\infty & \text{if } I_1 < |d_{\mathrm{I}}| + |d_{\mathrm{II}}| + |d_{\mathrm{III}}|, \end{cases} \tag{28}$$

where d_i, $i = \mathrm{I, II, III}$, are the eigenvalues of the strain rate tensor and where $H = k_{(\mathrm{MC})}\,\mathrm{cotg}\varphi$ is the matrix strength under hydrostatic tension. In the case of discontinuous velocity fields, eqn (28) takes the form

$$\pi(\underline{V}, \underline{n}) = \begin{cases} H\underline{V}\cdot\underline{n} & \text{if } \dfrac{\underline{V}\cdot\underline{n}}{|\underline{V}|} \geqslant \sin\varphi \\ +\infty & \text{if } \dfrac{\underline{V}\cdot\underline{n}}{|\underline{V}|} < \sin\varphi. \end{cases} \tag{29}$$

According to eqn (29), the admissible failure mechanisms are characterized by velocity vectors falling inside the cone of the outward normals to the Coulomb strength domain, with opening angle 90°–φ and axis coinciding with the normal $\underline{n}$ to the discontinuity surface.

If the composite is subjected to uniaxial stress $\underline{\underline{\Sigma}} = \Sigma \underline{e}_\vartheta \otimes \underline{e}_\vartheta$ at any orientation ϑ to the fibers, use of eqn (27′) in conjunction with eqn (6) leads to the following constrained maximization (or minimization) problem for the lower bounds on the macroscopic uniaxial tensile (Σ_0^+) or compressive (Σ_0^-) strength:

$$\Sigma_0^\pm(\vartheta) = \begin{cases} +\sup \\ -\inf_\sigma \end{cases} \left\{ \Sigma \,\middle|\, \sqrt{\left(\frac{\Sigma - \sigma}{2}\right)^2 + \Sigma\sigma s_\vartheta^2} \right.$$
$$\left. \leqslant \min\left\{ k_{(\mathrm{MC})}c_\varphi - \frac{\Sigma - \sigma}{2}s_\varphi;\;\; 2k_{(\mathrm{MC})}\mathrm{tg}(45^\circ \mp \varphi/2) \mp \frac{\Sigma - \sigma}{2} \right\}; \quad -\bar{\sigma}^- \leqslant \sigma \leqslant \bar{\sigma}^+ \right\},$$

where $c_\varphi = \cos\varphi$, $s_\varphi = \sin\varphi$. Solution of this problem gives

$$\Sigma_0^{\pm}(\vartheta) = \begin{cases} \bar{\sigma}^{\pm}\left(1-\dfrac{2s_\vartheta^2}{c_\varphi^2}\right) \mp 2k_{(\mathrm{MC})}\mu + \dfrac{2}{c_\varphi}\sqrt{k_{(\mathrm{MC})}^2+\bar{\sigma}^{\pm}s_\vartheta^2\left[\bar{\sigma}^{\pm}\left(\dfrac{s_\vartheta^2}{c_\varphi^2}-1\right)+2k_{(\mathrm{MC})}\mu\right]} \\ \qquad \text{if } \tilde{\sigma}^{\pm} \geqslant \bar{\sigma}^{\pm} \\ \dfrac{k_{(\mathrm{MC})}}{s_\vartheta(c_\vartheta \pm \mu s_\vartheta)} \quad \text{if } \tilde{\sigma}^{\pm} < \bar{\sigma}^{\pm} \quad \text{and} \quad \vartheta \leqslant 45^\circ \pm \varphi/2 \\ 2k_{(\mathrm{MC})}\mathrm{tg}(45^\circ \mp \varphi/2)\ (=\text{const.}) \quad \text{if } \vartheta \geqslant 45^\circ \pm \varphi/2 \end{cases} \tag{30a, b, c}$$

where $\mu = \mathrm{tg}\varphi$ and

$$\tilde{\sigma}^{\pm} = k_{(\mathrm{MC})}c_\varphi \frac{\pm s_\varphi + \sqrt{s_\varphi^2 + (s_\varphi^2 - c_\vartheta^2)/s_\vartheta^2}}{s_\varphi^2 - c_\vartheta^2}.$$

The upper bound on the macroscopic uniaxial strength given by mechanisms with oblique failure planes [see eqn (14)] is, in the present case:

$$\Sigma_\beta^{\mp}(\vartheta) = k_{(\mathrm{MC})} \frac{\pm s_\vartheta c_\beta \mu + \sqrt{1 - s_\vartheta^2 c_\beta^2}}{s_\vartheta c_\beta (s_\vartheta^2 c_\beta^2 / c_\varphi^2 - 1)}, \tag{31}$$

whereas the upper bound generated by mechanisms with planes perpendicular to $\underline{e}_y$ [$\Sigma_y^{\pm}(\vartheta)$, see eqn (13)] can be obtained as usual by putting $\beta = 0$ (i.e. $c_\beta = 1$) in the above eqn (31).

The results of a parametric study aimed at investigating the influence of φ on the bounds for the macroscopic uniaxial strength of the composite are presented in Fig. 6. The range within which the actual macroscopic strength falls is relatively well identified, since

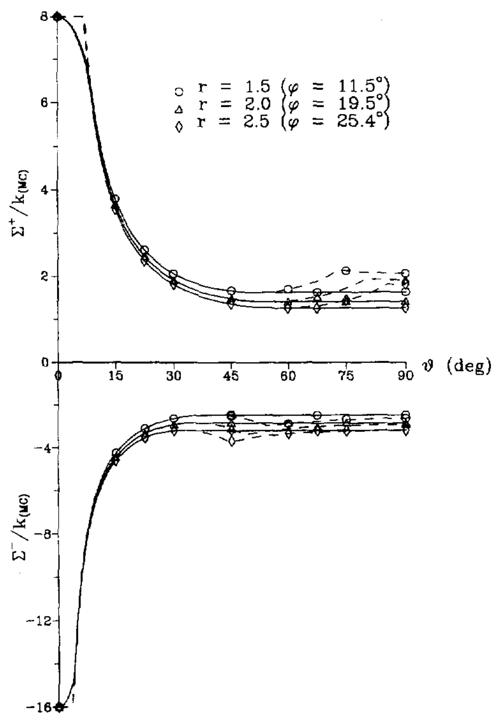

Fig. 6. Bounds on the uniaxial tensile/compressive strength of composites with matrix obeying the Mohr–Coulomb criterion for different ratios r between uniaxial tensile and compressive matrix strength. Hexagonal reinforcing array, $\eta = 0.65$, $\beta = 60°$. Solid lines are lower bounds, dashed lines are upper bounds.

the gap between lower and upper bounds is small for any considered value of the ratio $r = \sigma^-/\sigma^+$. Also, the mechanism with failure plane perpendicular to $\underline{e}_y$ yields upper bounds coinciding with the lower bounds of eqn (30b); this means that a certain range of orientation ϑ exists in which the macroscopic strength is exactly determined, in the spirit of limit analysis. It must be noted, however, that the lower bound curves do not exhibit any minimum at $\vartheta < 90°$, which is contrary to experimental observations, at least as regards the compressive strength (see e.g. the results of Tsai, 1968; Kim, 1981; Boehler and Delafin, 1982; and the following Figs 11–13), and which is predicted by parabolic or Drucker–Prager type matrices.

In particular, the macroscopic strength measured transverse to the fibers ($\vartheta = 90°$) is bounded by the values

$$\Sigma_0^{\pm}(90°) = 2k_{(\mathrm{MC})}\mathrm{tg}(45° \mp \varphi/2) \quad \text{and} \quad \Sigma_\beta^{\pm}(90°) = \frac{k_{(\mathrm{MC})}}{c_\beta(s_\beta \pm c_\beta \mu)}.$$

It should be noted that the condition $\varphi > \beta$ must be fulfilled in order to get significant upper bounds on a compressive stress Σ normal to the fibers; this condition ensures that the work done by the "external load" Σ in a mechanism with oblique failure plane is positive.

3.4. *General remarks on the results obtained without account of an interface criterion*

With reference to the upper bounds on the macroscopic uniaxial strength of composites with perfect bonding between fibers and matrix, derived in the previous sections, some remarks need to be made. In the case of composites with a parabolic-type matrix, Section 3.1, any mechanism characterized by *separation* of the parts that the unit cell is split into by the failure plane is admissible; for different $\underline{V}$ of this kind, both the dissipated power, π_{m}, and the work of the "external loads", $\underline{\underline{\Sigma}} \cdot \underline{n} \cdot \underline{V}$, differ, yielding upper bounds that can be more or less in agreement with the corresponding lower bounds. On the contrary, in the case of matrices obeying the Drucker–Prager or Coulomb criterion (Sections 3.2 and 3.3), the dissipated power is the same for any admissible $\underline{V}$; thus, the more degrees of freedom one has in the choice of $\underline{V}$ (i.e. in the maximization of the work of the external loads), the stricter the upper bounds become. The class of the admissible failure mechanisms depends upon the matrix strength parameters and the geometry of the reinforcing array. In fact, the smaller α and φ are for composites with Drucker–Prager and Coulomb matrices, respectively, the wider the cone of the admissible velocity vectors is, and the greater the scalar product $\underline{\underline{\Sigma}} \cdot \underline{n} \cdot \underline{V}$ can be made. It may happen that, for certain kind of failure mechanisms, no $\underline{V}$ falling within the cone of the admissible values can be associated with positive work of the external loads; this happens, for instance, in the case of a square reinforcing array ($\beta = 45°$), when the mechanism with oblique failure plane is considered: if the matrix obeys the Coulomb criterion, no significant upper bound for the compressive strength normal to the fibers is obtained if $\varphi \geqslant 45°$. These remarks explain the increase with α in the gap between the bounds in compression for Drucker–Prager matrices (Fig. 5); in the case of Coulomb-type matrices, this event is not evident in Fig. 6 only because φ takes small values in the explored range of ratios r. This problem is much less evident in tension, since the velocity vector is generally almost perpendicular to the failure plane.

The influence of the nature of the matrix on the bounds for the uniaxial strength can be assessed by comparing the curves plotted in Figs 4–6 for a given ratio $r = \sigma^-/\sigma^+$. For the sake of comparison, the uniaxial strength values of the fiber and matrix were taken as equal in all the three considered materials and the macroscopic strength was normalized by the same parameter, $k_{(\mathrm{MC})}$. Consequently, one may check that the lower bound curves exhibit the same minima, which coincide with the tensile or compressive strength of the unreinforced matrix. The orientations at which these minima are attained can be different, according to the type of strength criterion used for the matrix; recalling that $\tilde{\sigma}$ is related to the stress in the fibers, these orientations can be obtained by setting $\tilde{\sigma} = 0$ in eqns (19) or (25), whereas, in the case of Coulomb-type matrices, the macroscopic strength is constant

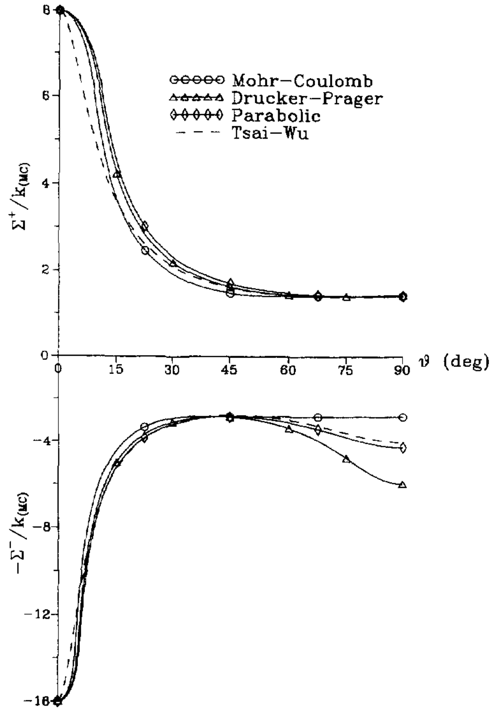

Fig. 7. Lower bounds on the uniaxial strength of composites with matrices obeying different strength criteria compared with the Tsai–Wu criterion.

between $45^\circ \pm \varphi/2$ and 90° and is given by eqn (30c). This comparison also shows that the best agreement between the lower and upper bounds is apparently achieved with a Coulomb-type matrix.

Finally, it seems interesting to compare the results given by the present model with the results yielded by the Tsai–Wu criterion, one of the most popular strength criteria for composites. Under plane stress conditions, this criterion is expressed by (see e.g. Tsai and Hahn, 1980)

$$\frac{\sigma_x^2}{X'X}+\frac{\sigma_y^2}{Y'Y}+2F_{xy}\sigma_x\sigma_y+\left(\frac{\sigma_{xy}}{S}\right)^2+\frac{X'-X}{X'X}\sigma_x+\frac{Y'-Y}{Y'Y}\sigma_y\leqslant 1, \tag{32}$$

where X (respectively X') is the tensile (respectively compressive) strength along the fiber direction, Y (respectively Y') is the tensile (respectively compressive) strength transverse to the fiber, S is the pure shear strength along the axes x, y and F_{xy} is a strength parameter related to the biaxial strength of the composite in the (x, y) plane. A total of six strength parameters are thus required to describe the in-plane composite strength.

In Fig. 7 a comparison was made of the lower bounds formulated here with different matrices with Tsai and Wu predictions; the minima in Tsai–Wu curves were identified with the uniaxial strengths of the unreinforced matrix to be adopted when the present model is applied. The close agreement of the different curves, expecially under uniaxial tension, has to be noted. No particular explanation should be sought for the fact that the Tsai–Wu curve in compression seems to exactly match the lower bound obtained with a parabolic-type matrix, since this mostly depends on the choice made for some of the six strength parameters.

On account of the agreement of the results obtained here with the Tsai–Wu criterion, which is widely acknowledged as adequate for the description of a number of experimental failure tests, one might suggest the use of the present model (with any one of the considered matrix strength criteria) for the characterization of experimental results. Actually, if one attempts to identify the model parameters using the results obtained on composite specimens, the risk arises of obtaining values without physical meaning, at least if the composite behaviour is markedly unequal in tension and compression. One reason for that may be the assumption of perfect bonding between fibers and matrix that has been employed until now, which may not allow the interpretation of certain aspects of the composite failure. Another argument supporting the introduction of an interface strength criterion is the fact that the failure mechanisms formulated in the preceding sections rely on the possibility of drawing failure planes through the matrix, which bars the applicability of the obtained upper bounds to composites with high volume fraction; actually, for this class of fiber-reinforced composites failure mechanisms are likely to involve the fiber–matrix interface also. The following section will be devoted to showing how the previously formulated lower and upper bounds change as a consequence of the introduction of an interface failure criterion.

3.5. *Influence of a Mohr–Coulomb type interface*

The fiber–matrix interface strength properties are accounted for by means of a Mohr–Coulomb type criterion, as suggested for instance by Outwater (1956) and Aboudi (1989). This criterion is given by

$$|\tau| \leqslant -\sigma \mathrm{tg}\varphi_{\mathrm{int}} + k_{\mathrm{int}} \tag{33}$$

[see also eqn (27)], where φ_{int} and k_{int} are the friction angle and the pure shear strength (cohesion) of the interface; these coefficients are to be interpreted as empirical factors that encompass all the chemical–physical interactions between the phases (DiLandro and Pegoraro, 1987).

Equation (33) defines a domain, g_{int}, in the space R^3 of the components of the stress vector $\underline{T}(\underline{n})$ acting at any point of the fiber–matrix interface. The support function of this domain, $\pi_{\mathrm{int}}(\underline{V};\underline{n})$, is obtained by replacing H in eqn (29) with $H_{\mathrm{int}} = k_{\mathrm{int}}\mathrm{cotg}\varphi_{\mathrm{int}}$ = interface strength under hydrostatic tension.

The contribution of the limited strength of the interface to the lower bound on the uniaxial macroscopic strength is accounted for by the domain G_{int}, eqn (5). This lower bound was expressed in Section 2.1 as

$$\Sigma^{\pm}_{0,\mathrm{int}} = \min\{\Sigma^{\pm}_0, \Sigma^{\pm}_{\mathrm{int}}\},$$

where $\Sigma^{\pm}_0$ depends on the matrix strength criterion and was computed in Sections 3.1–3.3. $\Sigma^{\pm}_{\mathrm{int}}$ can be obtained by specializing eqn (7) to the Coulomb criterion, which yields

$$\Sigma^{+}_{\mathrm{int}}(\vartheta) = \sup_{0 \leqslant v \leqslant 2\pi} \{\Sigma \,|\, |\tau| + \sigma\mu_{\mathrm{int}} \leqslant k_{\mathrm{int}};\ |\tau|^2 = \Sigma^2 s_\vartheta^2 c_v^2(1 - s_\vartheta^2 c_v^2);\ \sigma = \Sigma s_\vartheta^2 c_v^2\}, \tag{34}$$

where $\mu_{\mathrm{int}} = \mathrm{tg}\varphi_{\mathrm{int}}$, $c_v = \cos v$. The bound in compression, $\Sigma^{-}_{\mathrm{int}}$, is given by $-\inf$ of the above expression. Solution of this constrained maximization (or minimization) problem gives

$$\Sigma^{\pm}_{\mathrm{int}}(\vartheta) = \begin{cases} \dfrac{k_{\mathrm{int}}}{s_\vartheta(c_\vartheta \pm \mu_{\mathrm{int}} s_\vartheta)} & \text{if } \vartheta \leqslant 45^\circ \pm \varphi_{\mathrm{int}}/2 \\ \dfrac{2k_{\mathrm{int}}}{\mathrm{tg}(45^\circ \pm \varphi_{\mathrm{int}}/2)} (= \text{const.}) & \text{if } \vartheta \geqslant 45^\circ \pm \varphi_{\mathrm{int}}/2 . \end{cases} \tag{35}$$

The influence of the interface on the upper bounds for the macroscopic strength will be illustrated with reference to composites with matrices obeying the Coulomb criterion. First of all, it must be acknowledged that the interface may or may not be involved in failure mechanisms, depending on the geometry of the reinforcing array, as discussed in Section 2.2. Consider first a composite with rectangular reinforcing array, that fails according to a mechanism of the type described in Section 2.2, with the flat part of the failure

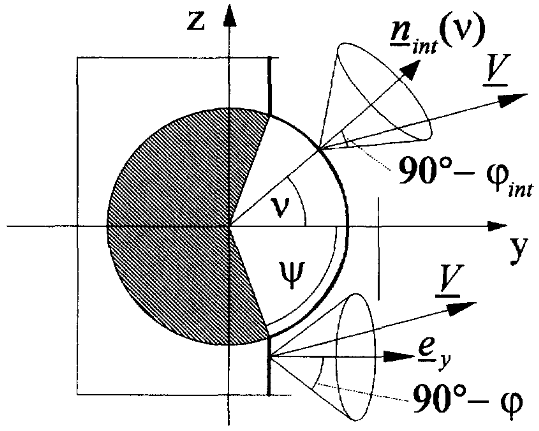

Fig. 8. Admissibility conditions for the failure velocity vector $\underline{V}$ in a mechanism involving the fiber–matrix interface.

surface perpendicular to $\underline{e}_y$ (see Fig. 2b). In this class of failure mechanisms, the angle ψ depicted in Fig. 2b and characterizing the mechanism which furnishes the strictest upper bound has to be computed:

$$\Sigma_y^{\pm}(\vartheta) = \sup_{\underline{V}} \Big\{ \Sigma \,|\, \Sigma \underline{e}_\vartheta \otimes \underline{e}_\vartheta : \ \underline{\underline{D}} \leqslant \min_{\psi} \pi_m(\underline{V};\underline{e}_y)\,|S_m(\psi)| + \int_{-\psi}^{+\psi} \pi_{int}(\underline{V};\underline{n}_{int}(v)) R \,dv;$$

$$\underline{\underline{D}} = 1/2(\underline{e}_y \otimes \underline{V} + \underline{V} \otimes \underline{e}_y), \quad \pi_m = H \underline{V} \cdot \underline{e}_y, \quad \pi_{int} = H_{int} \underline{V} \cdot \underline{n}_{int}(v);$$

$$\underline{V} \cdot \underline{e}_y \geqslant |\underline{V}| \sin\varphi; \quad \underline{V} \cdot \underline{n}_{int}(v) \geqslant |\underline{V}| \sin\varphi_{int}\ \forall v \in [-\psi, +\psi] \Big\}. \tag{36}$$

The constraints in eqn (36) mean that those velocity vectors $\underline{V}$ are admissible which fall both within the cone with axis $\underline{e}_y$ and opening angle $90° - \varphi$, and within all the cones with axis coinciding with the outward normal to the interface, $\underline{n}_{int}(v)$, and opening angle $90° - \varphi_{int}$ (see Fig. 8). Since upper bounds on the uniaxial strength in the (x, y) plane are being sought, only velocity vectors with $V_z = 0$ are considered in present calculations.

If the fiber radius R is expressed in terms of the geometrical features of the reinforcing array η, β, and the integral in eqn (36) is computed as

$$\int_{-\psi}^{+\psi} \pi_{int}(\underline{V};\underline{n}_{int}(v))\,dv = 2H_{int} \sin\psi\, V_y,$$

solution of the min–max problem, eqn (36), furnishes

$$\Sigma_y^{\pm}(\vartheta) = H \frac{\pm \mu_{int}^2 s_\vartheta - \mu \sqrt{c_\vartheta^2 + \gamma^2 (s_\vartheta^2 \mu_{int}^2 - c_\vartheta^2)/(1+\mu_{int}^2)}}{(s_\vartheta^2 \mu_{int}^2 - c_\vartheta^2) s_\vartheta}, \tag{37}$$

where

$$\gamma = 2\sqrt{\frac{\eta \,\mathrm{tg}\beta}{\pi}} \left(1 - \frac{H_{int}}{H}\right). \tag{38}$$

It could be easily proved that the power dissipated by this kind of mechanism is greater than its counterpart obtained when the interface is not involved (as in Section 3.3) if

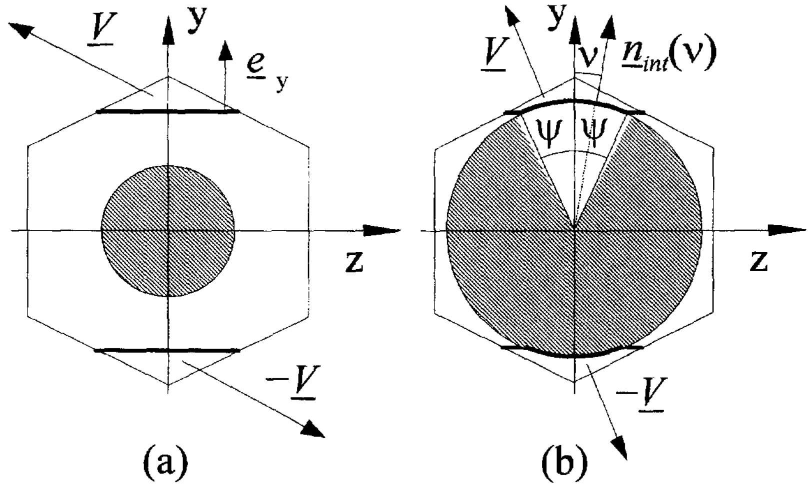

Fig. 9. Failure mechanisms of an hexagonal unit cell: (a) planes crossing the matrix, and (b) surface involving the fiber–matrix interface.

$H_{\text{int}} > H$; this condition amounts to saying that the interface is "stronger" than the matrix, so that for the composite it is more convenient to fail by excluding the interface. In this case, the upper bound computed in Section 3.3 is stricter than the one derived here.

Consider now a composite with hexagonal reinforcing array that fails according to the same type of mechanism as above. As mentioned in Section 2.2, if the fiber volume fraction is less than $\pi/8\text{tg}\beta$, the failure surface is a plane crossing the matrix only (see Fig. 9a) and the upper bound $\Sigma_y^{\pm}$ is given by eqn (31) with $c_\beta = 1$. For greater fiber volume fractions, part of the interface is involved in the failure surface (see Fig. 9b); the angle 2ψ subtended by this part can be computed based on purely geometrical considerations and is given by

$$2\psi = 2\arccos\sqrt{\frac{\pi\text{tg}\beta}{8\eta}}.$$

Setting again $V_z = 0$, the upper bound on the uniaxial strength turns out to be

$$\Sigma_y^{\pm}(\vartheta) = \frac{H(1-\gamma s_\psi)}{s_\vartheta\left(s_\vartheta \pm c_\vartheta\sqrt{c_\psi^2/\mu_{\text{int}}^2 - s_\psi^2}\right)}, \tag{39}$$

with $s_\psi = \sin\psi$, $c_\psi = \cos\psi$ and γ given by eqn (38). Note that, if the interface friction angle is too large, no vector $\underline{V}$ may be compatible with the constraints imposed by the interface strength criterion; no significant upper bound would then be obtained for this class of mechanisms, which corresponds to a negative argument of the square root in eqn (39).

When mechanisms with oblique failure planes are considered, both for rectangular and hexagonal reinforcing arrays, it is the cell geometry that dictates whether the interface is part of the failure surface. If the fiber volume fraction is less than the values listed in Section 2.2, the bounds obtained in Section 3.3 hold. For greater volume fractions, the upper bounds given by this kind of mechanism are obtained by computing

$$\Sigma_\beta^{\pm}(\vartheta) = \sup_{\underline{V}}\left\{\Sigma \,|\, \Sigma \underline{e}_\vartheta \otimes \underline{e}_\vartheta : \quad \underline{\underline{D}} \leqslant \pi_{\text{m}}(\underline{V};\underline{n}_\beta)\,|S_{\text{m}}| + \int_{-\psi}^{+\psi} \pi_{\text{int}}(\underline{V};\underline{n}_{\text{int}}(v))R\,\text{d}v;\right.$$

$$\underline{\underline{D}} = 1/2(\underline{n}_\beta \otimes \underline{V} + \underline{V} \otimes \underline{n}_\beta), \quad \pi_{\text{m}} = H\underline{V}\cdot\underline{n}_\beta, \quad \pi_{\text{int}} = H_{\text{int}}\underline{V}\cdot\underline{n}_{\text{int}}(v);$$

$$\left.\underline{V}\cdot\underline{n}_\beta \geqslant |\underline{V}|\sin\varphi;\ \underline{V}\cdot\underline{n}_{\text{int}}(v) \geqslant |\underline{V}|\sin\varphi_{\text{int}}\forall v \in [-\psi, +\psi]\right\}. \tag{40}$$

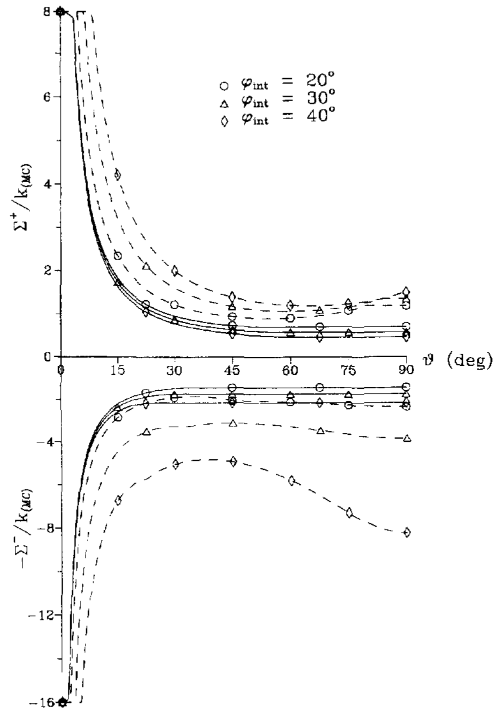

Fig. 10. Bounds on the uniaxial tensile/compressive strength of composites with matrix and interface obeying the Mohr–Coulomb criterion, for different interface friction angles, φ_{int}. Hexagonal reinforcing array, $\eta = 0.7$, $\beta = 60°$. Solid lines are lower bounds, dashed lines are upper bounds.

In this case, no symmetry consideration can be made, so that $\underline{V}$ is characterized by three nonvanishing components. The solution to this constrained maximization problem is reported in Appendix 2, together with the expression for the angle ψ.

All the above procedures are applicable to composites with matrices obeying different strength criteria. In particular, the upper bounds obtained in this section can be extended to matrices obeying the Drucker–Prager criterion, provided that φ and $k_{(MC)}$ are defined as

$$\sin\varphi = \frac{3\alpha}{\sqrt{1-3\alpha^2}}; \quad k_{(MC)} = \frac{k_{(DP)}}{\sqrt{1-12\alpha^2}}.$$

More involved calculations are required in the case of parabolic-type matrices, so that the solution of the min–max problems yielding the upper bounds on the uniaxial strength of the composite must be sought numerically.

Finally, the influence of the interface strength parameters on the bounds to the macroscopic strength was investigated for composites with Coulomb-type matrices, $k_{int} = 0.5k_{(MC)}$, $\varphi = \arcsin 1/3$ and variable φ_{int}. The results are shown in Fig. 10: it can be seen that the differences between the upper bounds and the corresponding lower bounds increase with increasing φ_{int}; this can be explained through arguments similar to those mentioned in Section 3.4, i.e. with increasing φ_{int} the range of the velocity vectors corresponding to admissible failure mechanisms becomes more limited, which leads to poorer upper bounds.

4. COMPARISON WITH EXPERIMENTAL DATA

In order to check the effectiveness of the proposed model, theoretical predictions will now be compared with data of experimental failure tests on composite specimens available in the literature. As shown in the previous sections, the model is completely defined by the strength parameters of the phases and the geometry of the reinforcing array: if these data are known, the macroscopic experimentally measured composite strength can be estimated.

Here a problem arises, since only the global strength values referred to the composite are usually reported in the literature; other data, such as the fiber volume fraction and the exact nature of the phases, are only seldom quoted by the experimenters. As a consequence, the model parameters have to be estimated through the model itself; comparison of these values with strength data relative to the phases, when available in the literature, can be used to check the reliability of the estimated values.

If perfect bonding between fiber and matrix cannot be assumed, six strength parameters have to be estimated. These can be found by making reference to the four tests that are simplest to perform (i.e. uniaxial tension/compression tests along the fibers and transverse to the fibers), in addition to other tests that will be discussed later. Incidentally, note that the number of parameters entering the model is equal to the number of parameters involved in the Tsai–Wu criterion, eqn (32), when uniaxial tests are dealt with; also note that, if the present model were extended to a general three-dimensional case, the number of involved parameters would remain unchanged, whereas the Tsai–Wu criterion would require nine strength parameters.

An additional problem related to the evaluation of the strength parameters comes from the fact that the present model provides bounds on the macroscopic strength, and not exact values. Which bounds then have to be employed in the evaluation of the model parameters? Here it is suggested that one assumes that the interface-controlled strength affects only the ultimate *tensile* behaviour of the composite. The reliability of this assumption can be checked *a posteriori* through comparison with experimental data; however, it is supported by the numerical findings of Yeh (1992), who performed finite element analyses of unit cells accounting for the tensile finite strength of the interface and observed that this strength has a more pronounced influence on the macroscopic tensile strength than on the compressive strength, at least as regards the composite strength transverse to the fibers.

In Sections 3.1–3.3, it was shown that lower and upper bounds are relatively close if perfect bonding between fiber and matrix is assumed, at least when η is not excessive and failure planes cutting only the matrix can be drawn. This means that either one of the bounds can be employed in identifying the model parameters in compression; since the lower bound is of greater simplicity, this latter one will be used. Thus, the two matrix strength parameters are evaluated on the basis of the composite strength in compression at $\vartheta = 90^\circ$ and at another orientation; experimental strength plots show a minimum at about $\vartheta = 60^\circ$: based on this minimum, model parameters are identified that give theoretical curves in close agreement with the entire set of experimental data in compression. Note that the fiber volume fraction is not required to be known for the identification of these first two parameters.

Once the matrix parameters are evaluated, the uniaxial fiber strengths can be found based on the composite strength in uniaxial tension/compression at $\vartheta = 0^\circ$: supposing that the fiber volume fraction, η, is known, eqn (12) yields

$$\sigma_{\mathrm{f}}^{\pm} = \frac{\Sigma^{\pm}(0^\circ) - (1-\eta)\sigma_{\mathrm{m}}^{\pm}}{\eta}.$$

The interface parameters are still left to be determined; to this end, use is made of experimental data in tension. This requires some care, since in Section 3.5 a certain discrepancy between lower and upper bounds for composites with interfaces of limited strength was shown to exist (see Fig. 10). Here it is proposed to employ the theoretical lower bound in tension also: an argument supporting this choice is that the lower bound comes from a static approach, in which a limit state for the composite is reached when the stress at any one point of the interface becomes critical. Due to the brittleness of the fiber–matrix interface in tension, this occurrence has similarities with the real behaviour of real polymeric composites, which is, as a consequence, probably closer to what is predicted by the theoretical lower bound rather than by the upper bound. This is further corroborated by the fact that, as ϑ increases, the upper bound curve attains a minimum and then increases as ϑ

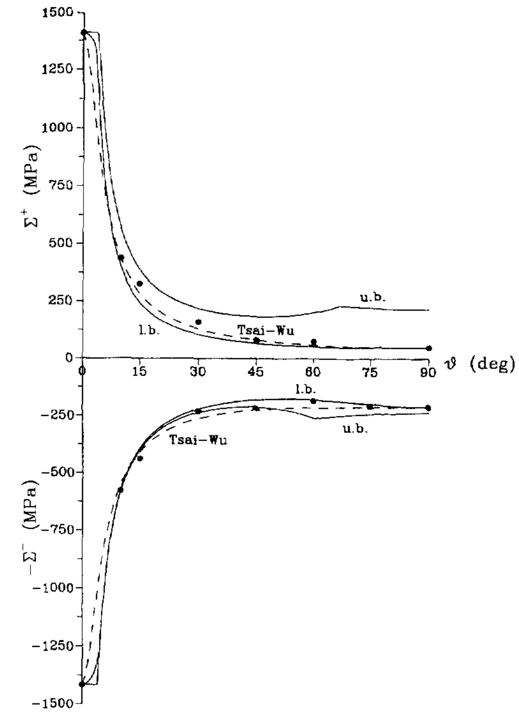

Fig. 11. Uniaxial failure tests on AS/3501 graphite–epoxy: experimental data after Tsai (1968) compared with the predictions of the present model (solid lines) and the Tsai–Wu criterion (dashed lines)—$k = 98.5$ MPa, $\alpha = 0.0235$, $k_{int} = 89.4$ MPa, $\varphi_{int} = 60°$.

approaches 90° ; the trend of the experimental data in tension is not of this kind and is, in contrast, similar to the lower bound curve.

That being stated, and keeping in mind that φ_{int} must be great enough if the interface is not to be involved in modelling the composite strength in compression, a friction angle $\varphi_{int} \approx 60°$ was found to describe with sufficient accuracy the available experimental results as a whole. This value is in agreement with the findings of DiLandro and Pegoraro (1987), who computed interface friction angles ranging between 57° for polyetherimide matrix composites and 62° for polyethersulphone matrix composites. Assuming for φ_{int} a fixed value of 60°, only the interface cohesion k_{int} is left to be determined, based on the composite tensile strength transverse to the fibers, $\Sigma^+(90°)$:

$$k_{int} = 1/2\Sigma^+(90°)\mathrm{tg}(45° + \varphi_{int}/2)$$

$$(\text{if } \varphi_{int} = 60°) = \Sigma^+(90°)\left(1 + \frac{\sqrt{3}}{2}\right),$$

see eqn (35).

Figures 11–13 show comparisons of the predictions of the present model with the experimental findings of various authors (Tsai, 1968; Kim, 1981; Boehler and Delafin, 1982) on polymeric composites with different types of fibers. Since the experimenters do not report more exhaustive information, the reinforcing array was assumed to be a regular hexagon ($\beta = 60°$) and a fiber volume fraction $\eta = 0.65$ was assumed in calculations: this latter value is commonly encountered in the literature (see e.g. the ASM *Engineered Material Handbook*, 1987). The matrix was presumed to comply with the Drucker–Prager criterion. The model parameters were computed according to the previously discussed procedure, based on the lower bound predictions; the same parameters were also employed in plotting the upper bound theoretical curves. Comparison with the predictions of the Tsai–Wu criterion is included in the same figures for the sake of completeness. The agreement between theoretical and experimental results is satisfactory on the whole. The experimental data

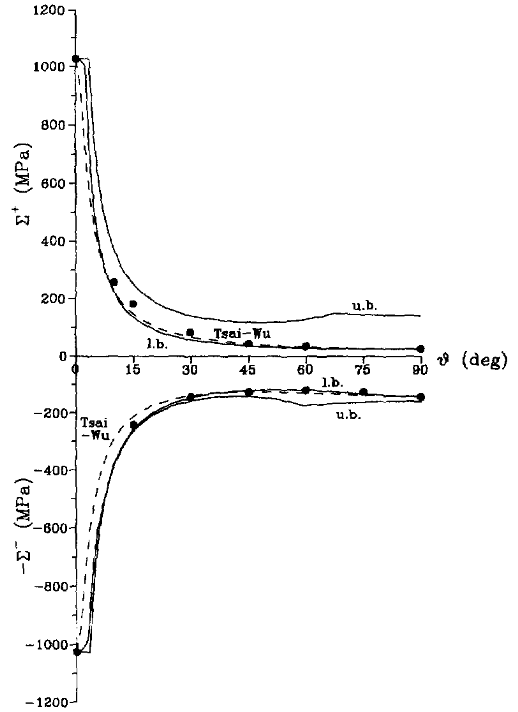

Fig. 12. Uniaxial failure tests on E-glass/epoxy: experimental data after Kim (1981) compared with the predictions of the present model (solid lines) and the Tsai–Wu criterion (dashed lines)—$k = 55.6$ MPa, $\alpha = 0.0293$, $k_{int} = 49.8$ MPa, $\varphi_{int} = 60°$.

points lie mostly within the band between the lower and the upper bound curves as they should do; the few exceptions to this rule are deemed to be attributable to scatter in the experimental data. The experimental data points of Figs 11 and 12 are also well matched by the Tsai–Wu criterion; the parameters defining the Tsai–Wu criterion could not be computed for the data of Fig. 13, since the experimenters do not give information regarding the tensile strength of the tested material.

The reliability of the procedure employed here for the identification of the model parameters is confirmed by the fact that the matrix strength parameters, reported in Figs 11–13, are consistent with the values available in the literature: for instance, the matrix pure shear strength turns out to range between 55 and 100 MPa for the materials considered here and similar values can be found for polymeric matrices e.g. in the ASM *Engineered Materials Handbook* (1987).

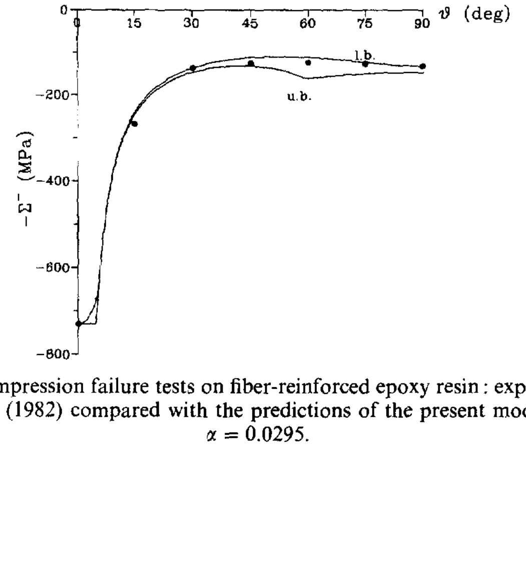

Fig. 13. Uniaxial compression failure tests on fiber-reinforced epoxy resin: experimental data after Boehler and Delafin (1982) compared with the predictions of the present model—$k = 59.7$ MPa, $\alpha = 0.0295$.

5. DISCUSSION AND CONCLUSIONS

Limit analysis theory, used in conjunction with a homogenization procedure, allowed the prediction of bounds on the uniaxial strength of polymeric-matrix composites on a macroscopic scale. The strength of the matrix phase is modelled using three different failure criteria and the influence of limited strength of the fiber–matrix interface is taken into account using a Mohr–Coulomb type strength criterion. The micromechanical approach is based on rather simple statically and kinematically admissible stress and strain rate fields, which allowed in most cases the derivation of analytical expressions for the bounds. Of course, more refined fields (suggested for example by finite element analyses) could be used, but the need of accounting for fields other than those used here is actually felt only for the cases where the gap between the computed bounds is not small.

Because of its own theoretical origin, the model presented here is capable of accounting only for dispersed failure modes. Indeed, since a unit cell is analysed, failure due to localized phenomena, often observed in composites with semi-brittle constituents, is precluded. This means that the model cannot be applied to determine the load-bearing capability of a composite laminate in the presence of delamination, possibly induced by edge effects, which can be one of the major causes of failure in composite structures. Also, the model cannot predict that for certain types of composites with brittle constituents an interface of limited strength actually increases the composite strength by blunting catastrophic crack growth, rather than decreasing it (see e.g. Cook and Gordon, 1964).

The above limitations, in addition to the assumptions employed in the backing-out procedure described in Section 4 to identify the strength parameters of the constituents, may give the impression of a rather speculative approach. It must be acknowledged that, if a phenomenological failure criterion were used (e.g. Tsai–Wu criterion), the problem of postulating *a priori* the failure mode to identify the model parameters would not arise. Actually, some of the assumptions made in the backing-out procedure can be justified by physical considerations and are in common with other authors; their reliability is further confirmed by the correspondence of the identified values with the strength parameters available in the literature for the constituents.

The authors do not intend to suggest the use of a specific strength criterion for polymeric matrix composites (although the use of the Drucker–Prager criterion in Section 4 allowed fitting of the experimental points reasonably well and might be an appropriate one for the polymers considered in the comparisons); of course, the choice of the appropriate criterion for the matrix phase depends on the material system studied. The criteria considered here were selected because they are capable of accounting for the unequal behaviour of the material in tension/compression, which is one of the main characteristics of polymers; the use of some of them was also suggested by other authors based on the results of uni- and biaxial tests. A complete set of experimental multiaxial tests is required to define the actual failure domain of the matrix, which may turn out to be different from those considered here. It is worth emphasizing, however, that the present model allows us to account for strength criteria of any kind which are deemed more accurate in modelling the ultimate properties of the constituents. Indeed, one of the authors' major concerns was to show the flexibility of the model.

In light of the comparisons made in Section 4, no advantage may seem to come from using the present model rather than the Tsai–Wu criterion. Actually, the degree of complexity inherent in a micromechanical approach can be justified only if more accurate results are obtained in comparison with a phenomenological one. Thus, the authors do not suggest that the employed micromechanical approach is superior to the phenomenological (macromechanical) approach based on the Tsai–Wu failure criterion, although it is believed that under general three-dimensional stress conditions some advantages would come from using the present model, which involves fewer strength parameters than the Tsai–Wu criterion (see also Section 4). On the positive side, the limit analysis coupled with the micromechanical approach may, when properly applied and interpreted, give useful insight into the mechanisms controlling the strength behaviour of a composite (see also Section 2.3) by providing bounds on composite strength under the employed set of assumptions.

Since here only one mechanism was taken into account for a given failure mode, any deviation from these bounds observed in real materials may be useful in shedding light on the neglected failure mechanisms that may be operative. Conversely, any behaviour that falls between such bounds may potentially be used to isolate the operative failure mechanisms.

Finally, mention should be made of the fact that the present model allows, in principle, the design of new composites with prescribed strength properties by properly selecting the constituents and their volume fractions; this possibility is precluded by the use of phenomenological criteria.

Since the present model appears to be promising in several respects, an extension to the simulation of the ultimate behaviour of composites subjected to more complex (multiaxial) stress states is planned. A first attempt in this direction was made by Taliercio (1992) for Drucker–Prager matrix composites, assuming perfect bounding between fiber and matrix.

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APPENDIX 1: DERIVATION OF THE SUPPORT FUNCTION OF THE PARABOLIC STRENGTH DOMAIN

Let G be a convex domain in R^N and let $\underline{x}, \underline{y}$ be two elements of R^N; a dot indicates inner product. Also, let $f(\underline{x})$ be the "gauge function" of G, i.e.

$$f(\underline{x}) \leqslant 0 \Leftrightarrow \underline{x} \in G.$$

The definition of "support function" $\pi(\underline{x})$ of G is (see e.g. Tyrrell Rockafellar, 1970)

$$\pi(\underline{x}) = \sup\{\underline{x}\cdot\underline{y} \mid \underline{y} \in G\}. \tag{A1}$$

The aim of this appendix is to show how the support function, eqn (16), of the parabolic strength domain defined by eqn (15) can be derived.

Since the constrained maximization problem, eqn (A1), has to be solved, use can be made of Lagrange's multiplier method. This amounts to finding the saddle point(s) of the function

$$L = -\underline{\underline{\sigma}} : \underline{\underline{d}} + \lambda f(\sigma),$$

where λ is a non-negative Lagrange's multiplier and $f(\sigma) = J_2' + 2/3aJ_1 - k_{(P)}^2$:

$$\pi(\underline{\underline{d}}) = -\inf_{\underline{\underline{\sigma}}} \sup_{\lambda \geqslant 0} L.$$

Stationariness of L with respect to λ yields the condition that the solution point $\underline{\underline{\sigma}}^*$ must be on the boundary of $G, f(\underline{\underline{\sigma}}^*) = 0$. Stationariness of L with respect to $\underline{\underline{\sigma}}$ yields

$$\underline{\underline{d}} = \lambda \frac{\partial f(\underline{\underline{\sigma}}^*)}{\partial \underline{\underline{\sigma}}} = \lambda[\underline{\underline{\sigma}}^* - 1/3(J_1^* - 2a)\underline{\underline{1}}], \tag{A2}$$

where $\underline{\underline{1}}$ is the identity tensor and J_1^* is the linear stress invariant computed at $\underline{\underline{\sigma}}^*$. With the aim of expressing π in terms of $\underline{\underline{d}}$ only, λ is eliminated from the problem by computing

$$I_1 = \mathrm{tr}\,\underline{\underline{d}} = 2\lambda a.$$

This shows that the requirement of non-negativeness for λ is equivalent to $I_1 \leqslant 0$; this condition is present in eqn (16).

Use of eqn (A2) and the condition $f(\underline{\underline{\sigma}}^*) = 0$ allows expression of $\underline{\underline{\sigma}}^*$ as a function of $\underline{\underline{d}}$:

$$\underline{\underline{\sigma}}^*(\underline{\underline{d}}) = \frac{2a}{I_1}\underline{\underline{d}} + \left[\frac{k_{(P)}^2}{2a} - 2a\left(\frac{1}{3} + \frac{I_2'}{I_1^2}\right)\right]\underline{\underline{1}}.$$

Finally, $\pi(\underline{\underline{d}})$ is computed as $\underline{\underline{\sigma}}^*(\underline{\underline{d}}) : \underline{\underline{d}}$ and eqn (16) is obtained.

APPENDIX 2. UPPER BOUNDS FOR MECHANISMS WITH OBLIQUE FAILURE SURFACE INVOLVING COULOMB-TYPE MATRIX AND INTERFACE

This appendix is devoted to the derivation of upper bounds on the macroscopic uniaxial strength for composites with a Coulomb-type matrix and interface, based on mechanisms with a failure surface having a flat part perpendicular to $\underline{n}_\beta$ and involving the interface. This requires solving the maximization problem in eqn (40), which involves long and tedious calculations without any conceptual difficulty; both for rectangular and hexagonal reinforcing arrays, these lead to a second order equation of the form

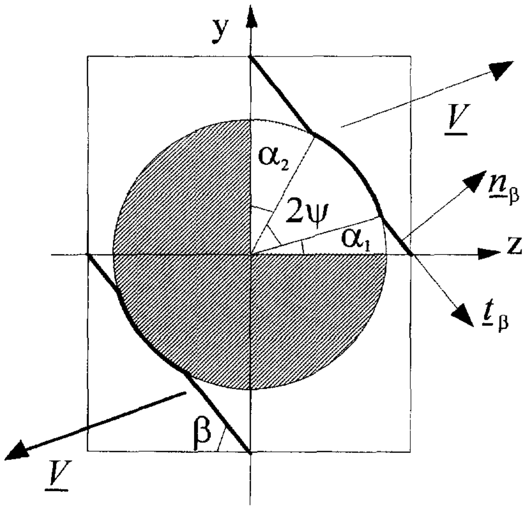

Fig. A1.

$$a_1\Sigma_{nn}^2 - (a_2\Sigma_{tn} + 2a_1a_4 - a_5a_2)\Sigma_{nn} + a_3\Sigma_{tn}^2 - (2a_5a_3 - a_4a_2)\Sigma_{tn} - \left(a_1a_3 - \frac{a_2^2}{4}\right)\Sigma_{xn}^2 + (a_5^2a_3 - a_5a_4a_2 + a_4^2a_1) = 0. \quad \text{(A3)}$$

Here Σ_{nn}, Σ_{tn} and Σ_{xn} are the components of the macroscopic stress referred to the axes (x, n_β, t_β) shown in Fig. A1; they are related to the macroscopic uniaxial stress Σ by the relationships

$$\Sigma_{nn} = \Sigma s_\beta^2 c_\beta^2; \quad \Sigma_{tn} = \pm\Sigma s_\beta^2 c_\beta s_\beta; \quad \Sigma_{xn} = \Sigma s_\beta c_\beta c_\beta.$$

The coefficients a_1 to a_5 are related to the strength properties of matrix and interface and to the geometry of the reinforcing array; in the case of rectangular reinforcing arrays, they are given by

$$a_1 = \frac{\cos^2(\alpha_1 + \beta)}{\sin^2\varphi_{\text{int}}} - 1; \quad a_2 = \frac{\sin 2(\alpha_1 + \beta)}{\sin^2\varphi_{\text{int}}}; \quad a_3 = \frac{\sin^2(\alpha_1 + \beta)}{\sin^2\varphi_{\text{int}}} - 1;$$

$$a_4 = H - 2\sqrt{\frac{\eta \text{tg}\beta}{\pi}}\, c_\beta\{2H\sin\psi - H_{\text{int}}[c_\beta(\cos\alpha_1 - \sin\alpha_2) - s_\beta(\sin\alpha_1 - \cos\alpha_2)]\};$$

$$a_5 = -2\sqrt{\frac{\eta \text{tg}\beta}{\pi}}\, c_\beta H_{\text{int}}[s_\beta(\cos\alpha_1 - \sin\alpha_2) + c_\beta(\sin\alpha_1 - \cos\alpha_2)].$$

Finally, α_1 and α_2 are the angles shown in Fig. A1 and depend on the geometry of the reinforcing array as follows:

$$\alpha_1 = 2\text{arctg}\left\{\frac{1 - \sqrt{1 + [1 - \pi\,\text{cotg}\beta/(4\eta)]\text{tg}^2\beta}}{[1 + \sqrt{\pi/(4\eta\text{tg}\beta)}]\text{tg}\beta}\right\};$$

$$\alpha_2 = 2\text{arctg}\left\{\frac{1 - \sqrt{1 + [1 - \pi\text{tg}\beta/(4\eta)]\,\text{cotg}^2\beta}}{[1 + \sqrt{\pi\text{tg}\beta/(4\eta)}]\,\text{cotg}\beta}\right\}.$$

For hexagonal reinforcing arrays, in the equations for a_4 and a_5 η should be replaced with 2η, whereas in the equations for α_1 and α_2 4η should be replaced with 2η.

Solving eqn (A3) for Σ yields upper bounds both on the tensile and the compressive uniaxial strengths of the composite.